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An Axiomatic Approach to Arbitration and its Application in Bargaining Games*

Kang Rong

Abstract

We define an arbitration problem as the triplet of a bargaining set and the offers submitted by two players. We characterize the solution to a class of arbitration problems using the axiomatic approach. The axioms we impose on the arbitration solution are "Symmetry in Offers," "Invariance" and "Pareto Optimality." The key axiom, "Symmetry in Offers," requires that whenever players' offers are symmetric, the arbitrated outcome should also be symmetric. We find that there exists a unique arbitration solution, called the symmetric arbitration solution, that satisfies all three axioms. We then analyze a simultaneous-offer game and an alternating-offer game. In both games, the symmetric arbitration solution is used to decide the outcome whenever players cannot reach agreement by themselves. We find that in both games, if the discount factor of players is close to 1, then the unique subgame perfect equilibrium outcome coincides with the Kalai-Smorodinsky solution outcome.

KEYWORDS: arbitration problem, simultaneous-offer game, alternating-offer game, Kalai- Smorodinsky solution

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1 Introduction

Arbitration occurs when two players are unable to reach agreement with each other. In this paper, we formally define the arbitration problem as the triplet that consists of offers submitted by two players and their bargaining set. An arbitration solution outcome is a point in the bargaining set chosen by an arbitrator. In order to obtain the arbitration outcome, the arbitrator usually follows a certain arbitration procedure. In the literature, there are two well-know arbitration procedures. One is the rule of equally-split-the-difference between players' offers, and the other is the final-offer arbitration rule.[1]

In this paper, we will use the axiomatic approach (Nash, 1950; Kalai and Smorodinsky, 1975) to characterize the arbitration procedure. An advantage of the axiomatic approach is that, we don't need to characterize the detailed arbitration process. Instead, we propose several axioms that an arbitration procedure should satisfy and then find the arbitration solution that satisfies those axioms.

The key axiom we impose on the arbitration procedure is "Symmetry in Offers," which requires *fairness* in arbitration. More particularly, it requires that whenever the two players' offers are symmetric with each other, the arbitrated outcome should also be symmetric. "Symmetry in Offers" appears to be a strong rule in the sense that it does not require symmetry in the bargaining set. However, "Symmetry in Offers" is a natural rule given that the arbitrator should make a decision primarily based on players' offers, instead of the shape of the bargaining set. In addition, it is a simple rule because it does not require the arbitrator to calculate the entire shape of the bargaining set. The other two axioms, "Invariance w.r.t. Affine Transformation" and "Pareto Optimality" are self-evident. They require *invariance* and *efficiency* in arbitration respectively. We find that there is a unique arbitration solution that satisfies all the three axioms. We call this solution the *symmetric arbitration solution*. The symmetric arbitration solution has a simple graphical representation: for any given bargaining set and offers submitted by the two players, the symmetric arbitration solution outcome is the intersection point of the Pareto frontier of the bargaining set with the line joining the component-wise minimum and component-wise maximum of the offers. When the Pareto frontier of the bargaining set is linear, the symmetric arbitration solution coincides

[1] Final offer arbitration is a procedure in which the arbitrator must choose one of the players' offers as the arbitration outcome (Stevens, 1966).

with the rule of "equally splitting the difference." The symmetric arbitration solution solution is "superior" to the rule of "equally splitting the difference" in that when the Pareto frontier of the bargaining set is nonlinear, "equally splitting the difference" results in an *inefficient* outcome, while the symmetric arbitration solution results in an *efficient* outcome.

Although our focus is to use "Symmetry in Offers" to characterize the symmetric arbitration solution, it is possible for us to use a weaker version of "Symmetry in Offers" to characterize the symmetric arbitration solution. The weaker version of "Symmetry in Offers," called "Weak Symmetry in Offers," requires that the arbitration solution outcome be symmetric whenever players' offers are symmetric *and* the bargaining set is symmetric. We show that the symmetric arbitration solution is the only solution that satisfies "Weak Symmetry in Offers," "Invariance," "Pareto Optimality," and "Strong Monotonicity."

We then propose two bargaining games in which, whenever the players are unable to reach agreement, an arbitration stage is reached and the symmetric arbitration solution is used to decide the outcome. The first game is a simultaneous-offer game. In this game, two players make offers simultaneously. If the offers are compatible, then each player gets what he demands, otherwise the game moves to the arbitration stage. In the arbitration stage, the symmetric arbitration solution is utilized to determine the outcome. This game is similar to the second Nash demand game in Anbarci and Boyd (2011)[2]. Both games are variants of the Nash demand game (Nash, 1953) and have arbitration stages. The difference is that the game in Anbarci and Boyd (2011) uses the rule of "equally splitting the difference" at the arbitration stage, but our game uses the symmetric arbitration solution.

Our second game is an alternating-offer game. In this game, at stage 1, player 1 makes an offer and player 2 decides whether to accept or reject it. If player 2 chooses to reject the offer, then the game moves to the next stage, at which player 2 makes an offer and player 1 decides whether to accept or reject it. If player 1 rejects the offer, then the game moves to the arbitration stage in which the symmetric arbitration solution is used to decide the final outcome. This game can be regarded as a variant of the game proposed by

[2]The game considered in Anbarci and Boyd (2011) can be rephrased as follows. At the first stage, two players make offers simultaneously. If the offers are compatible, then each player gets what he demands, otherwise with probability $1-p$ the game terminates with the disagreement point as the outcome, and with probability p the game goes to the arbitration stage in which the rule of "equally splitting the difference" is used to decide the outcome. Note that the probability p of moving to the arbitration stage in their game is equivalent to the discount factor δ in our game when the disagreement point is normalized to $(0, 0)$.

Yildiz (2011) and the game studied by Rong (2012). In all of those games, two players make offers sequentially and if both offers are rejected, the game moves to an arbitration stage. Our game differs from Yildiz (2011) and Rong (2012) in that our game uses the symmetric arbitration solution at the arbitration stage, while both the game in Yildiz (2011) and the game in Rong (2012) use final offer arbitration.

In both the simultaneous-offer game and the alternating-offer game that we consider, the only arbitration cost is the time cost, which is measured by the common discount factor of players. Our equilibrium analyses show that, in both games, when the discount factor is close to 1 (i.e., the time cost is low), players tend to make extreme offers. The threshold discount factor required for players to make extreme offers is relatively small. In particular, when the Pareto frontier is linear, the threshold discount factor is $\frac{2}{3}$ for the simultaneous-offer game and is 0.91 for the alternating-offer game. In addition, we find that, when both players make extreme offers, the arbitrated outcome coincides with the Kalai-Smorodinsky solution outcome.

The result that as the discount factor becomes close to 1, the only equilibrium requires each player to make the extreme offer is not surprising. Actually, it is well known in the literature that if a bargaining process involves an arbitration mechanism which allows for compromise between offers, then the bargaining process is subject to the so-called *chilling effect* (Feuille, 1975; Deck and Farmer, 2007). That is, players tend to take extreme positions before arbitration. This tendency is stronger when players become more patient.

This paper is organized as follows. The next section is the axiomatic characterization of the arbitration problem. Section 3 presents the main result. Section 4 provides an alternative axiomatic characterization of the symmetric arbitration solution using the axiom of Weak Symmetry in Offers. Section 5 discusses the two bargaining games with symmetric arbitration, i.e., the "simultaneous-offer game with symmetric arbitration" and the "alternating-offer game with symmetric arbitration." Concluding remarks are offered in section 6.

2 Axiomatic Characterization of Arbitration Problem

Suppose there are two players who are expected utility maximizers. Let $S \subset R^2$ denote the bargaining set, which includes all possible bargaining outcomes, measured in expected utility level. Let $(x_1, y_1) \in S$ denote player 1's final offer submitted to an arbitrator and $(x_2, y_2) \in S$ denote player 2's final offer submitted to the arbitrator. Note we always use x to represent player 1's payoff and y to represent player 2's payoff.

We assume the bargaining set S is nonempty, convex, compact and strictly comprehensive. The definitions of "comprehensiveness" and "strict comprehensiveness" are given below:

Definition 1. *S is **comprehensive** if $\exists(d_1, d_2) \in R^2$ s.t. $\forall(x, y) \in S$, we have (i) $(x, y) \geq (d_1, d_2)$, and (ii) if $(d_1, d_2) \leq (x', y') \leq (x, y)$, then $(x', y') \in S$.*

Definition 2. *S is **strictly comprehensive** if S is comprehensive and for any $(x, y) \in S$ and $(x', y') \in S$ with $(x', y') \geq (x, y)$ and $(x', y') \neq (x, y)$, there exists a $(x'', y'') \in S$ such that $(x'', y'') >> (x, y)$.*

If we regard d as the disagreement point, then the "comprehensiveness" of a bargaining set simply requires: (i) for each player, the utility level at the disagreement point is the lowest possible utility level that he can get from bargaining; (ii) each player can freely dispose any utility that is higher than the disagreement point.

Strict comprehensiveness further requires the Pareto frontier of the bargaining set be strictly downward-sloping. We need a bargaining set to be strictly comprehensive to avoid the case that the Pareto frontier contains a flat or vertical segment. A typical strictly comprehensive bargaining set S is shown in Figure 1.

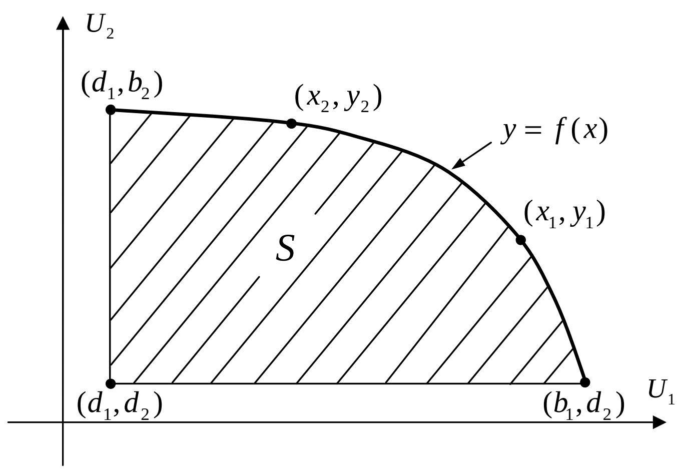

Figure 1: Bargaining set and players' offers.

Any nonempty bargaining set S that is convex, compact and strictly comprehensive determines a *unique* $d = (d_1, d_2)$ that satisfies Definition 1. We use $d(S)$ to denote this point.

The Pareto frontier of the bargaining set S is defined as $PF(S) = \{p \in S : q \geq p \text{ with } q \neq p \Rightarrow q \notin S\}$. We assume that each player can only make an offer *on* the Pareto frontier. This assumption is made for simplicity, although it is not essential for our main results.

Define $b_i = max\{U_i : (U_1, U_2) \in S\}$ as player i's maximal possible utility level from the bargaining set. Define the function $f : x \to max\{y|(x, y) \in S\}$ for $x \in [d_1, b_1]$. Thus $\{(x, f(x))|x \in [d_1, b_1]\}$ denotes the Pareto frontier. Our assumption that S is convex, compact and strictly comprehensive implies that f is a strictly decreasing function on $[d_1, b_1]$ with $f(d_1) = b_2$ and $f(b_1) = d_2$.

Now define $\Sigma = \{S \subset R^2 \,|S$ is nonempty, convex, compact, strictly comprehensive$\}$ and $\mathcal{B} = \{((x_1, y_1), (x_2, y_2), S) \,|(x_1, y_1) \in PF(S)\,,\ (x_2, y_2) \in PF(S),\ (x_1, y_2) \notin S \text{ and } S \in \Sigma\}$. We call any $((x_1, y_1), (x_2, y_2), S) \in \mathcal{B}$ an *arbitration problem.*[3] An *arbitration solution* is any function $g : \mathcal{B} \to R^2$ such that $g((x_1, y_1), (x_2, y_2), S) \in S$. We may write $g((x_1, y_1), (x_2, y_2), S) = (g_1((x_1, y_1), (x_2, y_2), S), g_2((x_1, y_1), (x_2, y_2), S))$, where $g_i((x_1, y_1), (x_2, y_2), S)$ is the arbitration outcome for player i.

We will propose the following three axioms that an arbitration solution should satisfy:

Definition 3. *An arbitration solution g is a* ***symmetric arbitration solution*** *if it satisfies the following three axioms:*

1. *Axiom 1 (Symmetry in Offers): For any arbitration problem $((x_1, y_1), (x_2, y_2), S) \in \mathcal{B}$ with $x_1 = y_2$ and $x_2 = y_1$, we have $g_1((x_1, y_1), (x_2, y_2), S) = g_2((x_1, y_1), (x_2, y_2), S)$.*

2. *Axiom 2 (Invariance w.r.t. Affine Transformation): If $A : R^2 \to R^2$ represents a strictly increasing affine transformation, i.e., $A(x, y) = (a_1 x + c_1, a_2 y + c_2)$ for some positive constant a_i and some constant c_i, then we have $g(A(x_1, y_1), A(x_2, y_2), A(S)) = A(g((x_1, y_1), (x_2, y_2), S))$ for any $((x_1, y_1), (x_2, y_2), S) \in \mathcal{B}$.*

3. *Axiom 3 (Pareto Optimality): For any arbitration problem $((x_1, y_1), (x_2, y_2), S) \in \mathcal{B}$, we have $g((x_1, y_1), (x_2, y_2), S) \in PF(S)$.*

Axiom 1 requires that if the offers from the two players are symmetric around the 45 degree line, then the arbitration solution outcome should also

[3] Notice that the arbitration problem we consider involves *incompatible* offers (i.e., $(x_1, y_2) \notin S$). If players' offers are compatible, then each player simply gets what he demands (and arbitration is not necessary). In addition, notice that our arbitration problem consists of two players' offers and a bargaining set, while the classic bargaining problem proposed by Nash (1950) consists of a disagreement point and a bargaining set.

be symmetric (i.e., on the 45 degree line). That is, if each player makes the same demand for himself and suggests the same payoff for his opponent, then the arbitrated outcome should result in the same payoff for each player. Axiom 1 does not require symmetry in the bargaining set. However, we still regard Axiom 1 as a natural rule for the following reasons. First, an arbitrator should primarily focus on the offers of players, instead of the shape of the bargaining set. Second, it is generally costly for the arbitrator to calculate the entire shape of the bargaining set. Axiom 1 (together with Axiom 2 and Axiom 3) only requires that the arbitrator calculate a fraction of the bargaining set in order to determine the arbitration outcome on the Pareto frontier.[4]

Axiom 2 is adapted from Nash (1950). The idea behind this axiom is that the arbitration outcome should only depend on players' underlying preferences and not on their utility representations. Hence, for two arbitration problems with the same preferences and the same physical offers submitted by the players, the arbitration outcome should also be the same (with correspondingly different utility representation). Note that players' utilities are expected utilities, so a player's utility is unique up to strictly increasing affine transformation. Finally, Axiom 3 simply requires the arbitration outcome to be efficient.

3 Main Result

It turns out the symmetric arbitration solution is unique and has a simple representation. For $p_1, p_2 \in R^2$, let $L(p_1, p_2)$ denote the line joining p_1 and p_2. We have the following result:

Theorem 1. *There is one and only one symmetric arbitration solution, denoted by γ. The function γ has the following simple graphic representation. For any arbitration problem $((x_1, y_1), (x_1, y_2), S) \in \mathcal{B}$, $\gamma((x_1, y_1), (x_2, y_2), S)$ is the intersection point of $L((x_1, y_1) \wedge (x_2, y_2), (x_1, y_1) \vee (x_2, y_2))$ with $PF(S)$ (see Figure 2).*

Proof: For a given arbitration problem $((x_1, y_1), (x_2, y_2), S) \in \mathcal{B}$, we have two cases:

(i) $y_2 > x_2$**.**

We need to find a strictly increasing affine transformation that transforms the given problem $((x_1, y_1), (x_2, y_2), S)$ to an offer-symmetric prob-

[4] I am indebted to Ichiro Obara and an anonymous referee for suggesting the above explanations for the axiom of Symmetry in Offers.

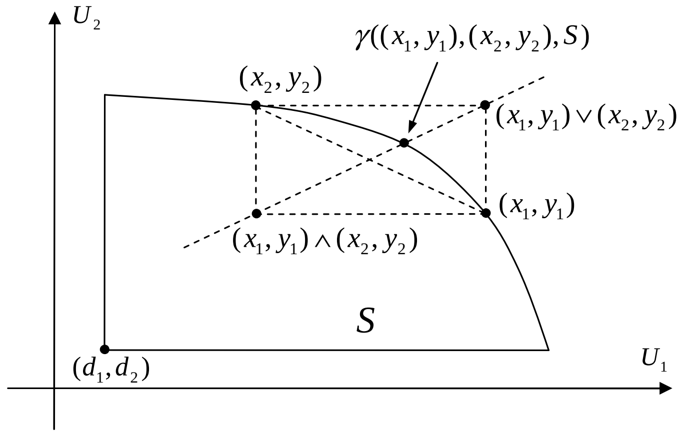

Figure 2: Symmetric arbitration solution.

lem $((x_1', y_1'), (x_2, y_2), S')$, where $x_1' = y_2$ and $y_1' = x_2$ (see Figure 3). Let $A_i^*(x) = a_i^* x + c_i^*$ $(i = 1, 2)$ be such a transformation. Then we have:

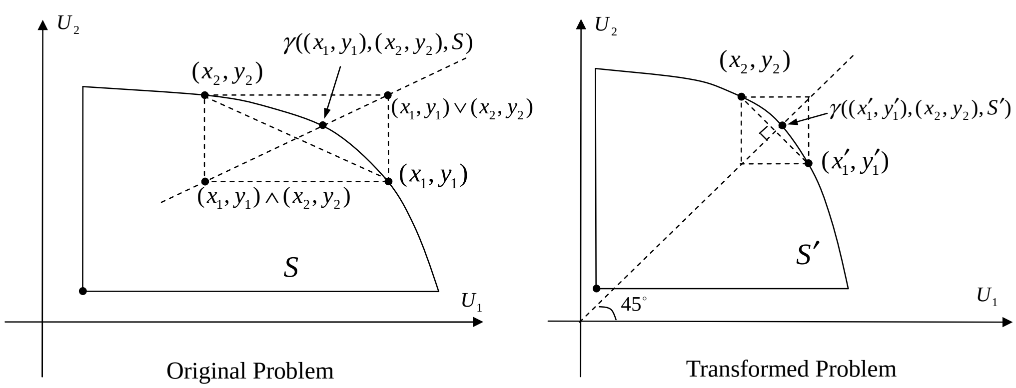

Figure 3: Transformation of the arbitration problem.

$$\begin{cases} x_2 = a_1^* x_2 + c_1^* \\ y_2 = a_2^* y_2 + c_2^* \end{cases} \text{and} \begin{cases} y_2 = a_1^* x_1 + c_1^* \\ x_2 = a_2^* y_1 + c_2^* \end{cases}. \tag{1}$$

Solving the equations, we have:

$$\begin{cases} a_1^* = \dfrac{y_2 - x_2}{x_1 - x_2} \\ a_2^* = \dfrac{x_2 - y_2}{y_1 - y_2} \end{cases} \text{and} \begin{cases} c_1^* = \dfrac{x_2(x_1 - y_2)}{x_1 - x_2} \\ c_2^* = \dfrac{y_2(y_1 - x_2)}{y_1 - y_2} \end{cases}. \tag{2}$$

Since $(x_1, y_1) \in PF(S)$, $(x_2, y_2) \in PF(S)$, $(x_1, y_2) \notin S$ and the Pareto frontier is strictly downward-sloping, we must have $x_1 > x_2$ and $y_1 < y_2$. Note we have also assumed that $y_2 > x_2$. It can be verified that $a_1^* > 0$ and $a_2^* > 0$, which ensures that the above affine transformation is indeed an expected utility transformation.

If (u_1^*, u_2^*) is the symmetric arbitration solution to the original arbitration problem $((x_1, y_1), (x_2, y_2), S)$, then by Axiom 2, $(a_1^* u_1^* + c_1^*, a_2^* u_2^* + c_2^*)$ is the symmetric arbitration solution to the transformed problem $((x_1', y_1'), (x_2, y_2), S')$. Since $((x_1', y_1'), (x_2, y_2), S')$ is symmetric in offers, the symmetric arbitration solution to it must be on the 45 degree line. Hence, we have:

$$a_1^* u_1^* + c_1^* = a_2^* u_2^* + c_2^*. \tag{3}$$

Using equations 2, equation 3 can be rewritten as:

$$u_2^* = \frac{y_2 - y_1}{x_1 - x_2} u_1^* + \frac{x_1 y_1 - x_2 y_2}{x_1 - x_2}. \tag{4}$$

It can be verified that the line

$$u_2 = \frac{y_2 - y_1}{x_1 - x_2} u_1 + \frac{x_1 y_1 - x_2 y_2}{x_1 - x_2}$$

is the line that passes through $(x_1, y_1) \wedge (x_2, y_2)$ and $(x_1, y_1) \vee (x_2, y_2)$. Now, by Axiom 3 (Pareto Optimality), we can conclude that (u_1^*, u_2^*) must be the intersection point of $L((x_1, y_1) \wedge (x_2, y_2), (x_1, y_1) \vee (x_2, y_2))$ with the Pareto frontier.

(ii) $y_2 \leq x_2$**.** We can always find a strictly increasing affine transformation such that the transformed arbitration problem has the property $y_2' > x_2'$. Then we go back to case (i) and the remaining proof is straightforward. □

A graphic representation of the symmetric arbitration solution is shown in Figure 2.

The idea of the proof is that, for any offer-nonsymmetric arbitration problem $((x_1, y_1), (x_2, y_2), S)$, we can always find a strictly increasing affine transformation to transform it to an offer-symmetric problem $((x_1', y_1'), (x_2, y_2), S')$, where $x_1' = y_2$ and $y_1' = x_2$ (see Figure 3). Due to the axiom of Pareto optimality and the axiom of Symmetry in Offers, the symmetric arbitration solution to the problem $((x_1', y_1'), (x_2, y_2), S')$ must be the intersection point of the 45 degree line with the Pareto frontier. Then, using the inverse of the above affine transformation, we can transform this solution outcome back to the original problem. It can be verified that the solution to the original problem is exactly the intersection point of $L((x_1, y_1) \wedge (x_2, y_2), (x_1, y_1) \vee (x_2, y_2))$ with $PF(S)$.

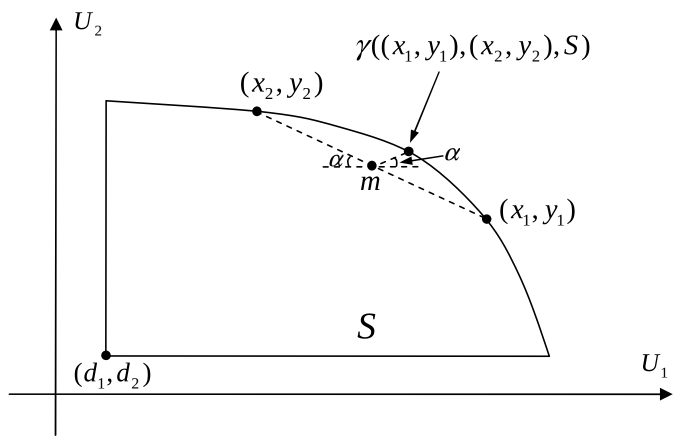

Figure 4: Another representation of the symmetric arbitration solution.

Another graphic interpretation of the solution is as follows. For the arbitration problem $((x_1, y_1), (x_2, y_2), S)$, connect the two offers (x_1, y_1) and (x_2, y_2) with a line and denote its middle point by m. Now, draw a line through m with a slope that is the negative of the slope of $L((x_1, y_1), (x_2, y_2))$. Then, the intersection point of this new line with the Pareto frontier is the symmetric arbitration solution (see Figure 4). The essential point here is that the line joining m and the solution point $\gamma((x_1, y_1), (x_2, y_2), S)$ always has a slope that is the negative of the slope of the line joining (x_1, y_1) and (x_2, y_2). Note this is true for any offer-symmetric arbitration problem because of the axiom of Symmetry in Offers. This is also true for any offer-nonsymmetric arbitration problem, because (i) any offer-nonsymmetric problem can be transformed to an offer-symmetric problem by some strictly increasing affine transformation, and (ii) two lines with slopes that are opposite in sign is a property preserved by any affine transformation.[5]

When the Pareto frontier of the bargaining set is linear, the symmetric arbitration solution outcome coincides with the outcome of "equally splitting the difference." When the Pareto frontier of the bargaining set is nonlinear, "equally splitting the difference" results in an *inefficient* outcome (point m in Figure 4), while the symmetric arbitration solution results in an *efficient* outcome.

[5]See Nash (1953) for a similar geometric explanation for the Nash bargaining solution.

4 Another Axiomatic Characterization of Symmetric Arbitration Solution[6]

In this section, we propose a weaker version of the axiom of Symmetry in Offers, called Weak Symmetry in Offers. It requires that the arbitration solution outcome be symmetric whenever players' offers are symmetric *and* the bargaining set is symmetric. It turns out that the symmetric arbitration solution is the unique arbitration solution that satisfies the following four axioms: Weak Symmetry in Offers, Invariance, Pareto Optimality, and Strong Monotonicity.

Definition 4. *Let g be an arbitration solution. The axiom of Weak Symmetry in Offers and the axiom of Strong Monotonicity are defined as follows:*

1. *Axiom $1'$ (Weak Symmetry in Offers): For any arbitration problem $((x_1, y_1), (x_2, y_2), S) \in \mathcal{B}$ where $x_1 = y_2$, $x_2 = y_1$ and S is symmetric, we have $g_1((x_1, y_1), (x_2, y_2), S) = g_2((x_1, y_1), (x_2, y_2), S)$.*

2. *Axiom 4 (Strong Monotonicity): For any two arbitration problems $((x_1, y_1), (x_2, y_2), S) \in \mathcal{B}$ and $((x_1, y_1), (x_2, y_2), S') \in \mathcal{B}$, if $S' \supset S$, then $g_1(((x_1, y_1), (x_2, y_2), S')) \geq g_1(((x_1, y_1), (x_2, y_2), S))$ and $g_2(((x_1, y_1), (x_2, y_2), S')) \geq g_2(((x_1, y_1), (x_2, y_2), S))$.*

Theorem 2. *The symmetric arbitration solution γ is the unique arbitration solution that satisfies Axiom $1'$, Axiom 2, Axiom 3, and Axiom 4.*

Proof: It is easy to verify that the symmetric arbitration solution γ satisfies Axiom $1'$, Axiom 2, Axiom 3, and Axiom 4. Now, assume that there is another arbitration solution μ that satisfies all the four axioms. We will show that $\mu((x_1, y_1), (x_2, y_2), S) = \gamma((x_1, y_1), (x_2, y_2), S)$ for any $((x_1, y_1), (x_2, y_2), S) \in \mathcal{B}$.

It is without loss of generality to assume that $y_2 > x_2$.[7] Similar to part (i) of proof of Theorem 1, we can find a strictly increasing affine transformation (A_1^*, A_2^*) such that $((x_1, y_1), (x_2, y_2), S)$ can be transformed to an offer-symmetric arbitration problem $((x_1', y_1'), (x_2, y_2), S')$, where $x_1' = y_2$ and $y_1' = x_2$. Figure 5 illustrates the transformed arbitration problem. Let $S'' =$ convex hull $\{(x_1', y_1'), (x_2, y_2), (x_2, y_1'), \gamma((x_1', y_1'), (x_2, y_2), S')\}$. Since S'' is symmetric, by Axiom $1'$, $\mu((x_1', y_1'), (x_2, y_2), S'') = \gamma((x_1', y_1'), (x_2, y_2), S')$. Since $S' \supset S''$,

[6] I am indebted to an anonymous referee who suggested that I use a weaker symmetry axiom and some type of monotonicity axiom to characterize the symmetric arbitration solution.

[7] If $y_2 \leq x_2$, then we can always transform the arbitration problem to a new problem, which has the property $y_2' > x_2'$.

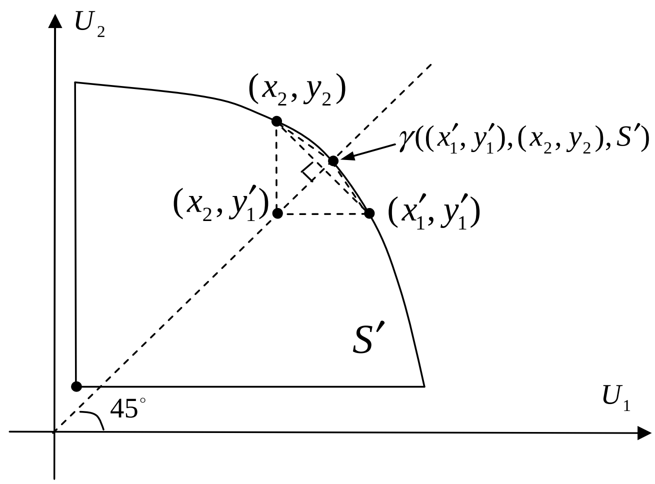

Figure 5: Transformed arbitration problem.

by Axiom 4, we must have $\mu((x'_1, y'_1), (x_2, y_2), S') = \mu((x'_1, y'_1), (x_2, y_2), S'')$, so, $\mu((x'_1, y'_1), (x_2, y_2), S') = \gamma((x'_1, y'_1), (x_2, y_2), S')$. Now, we can use the inverse of the transformation (A_1^*, A_2^*) to transform the solution $\mu((x'_1, y'_1), (x_2, y_2), S')$ back to the original problem, and we must have $\mu((x_1, y_1), (x_2, y_2), S) = \gamma((x_1, y_1), (x_2, y_2), S)$. □

5 Bargaining Games with Symmetric Arbitration

This section will analyze two bargaining games that involve symmetric arbitration. One is the simultaneous-offer game, and the other is the alternating-offer game.

From this point on, we fix the bargaining set S and we will simply write $\gamma((x_1, y_1), (x_2, y_2), S)$ as $\gamma((x_1, y_1), (x_2, y_2))$ whenever there is no confusion. We use $\delta \in (0, 1]$ to denote the discount factor, which means 1 unit of utility at the next stage is equivalent to δ unit of utility at the current stage. Finally, we assume throughout this section that $d(S) = (0, 0)$.

The following lemma states that a player's payoff obtained from the symmetric arbitration solution is strictly increasing in both his own demand and his opponent's suggested payoff for him. This implies that, if a player takes a stronger position (i.e., demand more) before arbitration, then he will get more payoff from arbitration.[8]

[8]This is true for any arbitration procedure that allows for compromises between offers.

Lemma 1. *For x_1, $x_2 \in [0, b_1]$, $\gamma_1((x_1, f(x_1)), (x_2, f(x_2)))$ is strictly increasing in x_1 and x_2; $\gamma_2((x_1, f(x_1)), (x_2, f(x_2)))$ is strictly decreasing in x_1 and x_2.*

Proof: see the appendix. □

5.1 Simultaneous-Offer Game

Simultaneous-offer game is also known as the Nash demand game. In the original Nash demand game (Nash 1953), two players make demands (offers) simultaneously. If their demands are compatible, then each player gets what he demands; otherwise, each player gets the disagreement payoff. One disadvantage of the Nash demand game is that it is a one-stage game that does not allow for renegotiation or arbitration. In the literature, many variants of the Nash demand game have been proposed to deal with this problem (e.g., Howard, 1992; Anbarci and Boyd, 2011).[9] Here, we are going to propose a new Nash demand game, in which players move to an arbitration stage whenever their offers are incompatible. In addition, we assume that the symmetric arbitration solution is used at the arbitration stage. In particular, we define the *simultaneous-offer game (Nash demand game) with symmetric arbitration* as follows:

1. Stage 1: player 1 and player 2 submit their offers simultaneously. Let $(x_1, y_1) \in PF(S)$ be the offer submitted by player 1 and $(x_2, y_2) \in PF(S)$ be the offer submitted by player 2. If (x_1, y_1) and (x_2, y_2) are compatible, then (x_1, y_2) is the outcome. Otherwise, the game moves to Stage 2.

2. Stage 2: an arbitrator decides the outcome using the *symmetric arbitration solution*, i.e., $\gamma((x_1, y_1), (x_2, y_2))$ is the arbitrated outcome.

Notice that players' payoffs obtained at stage 2 are discounted by δ. So, if the game moves to arbitration, the arbitrated payoffs received by players are $\delta\gamma((x_1, y_1), (x_2, y_2))$. Before characterizing the equilibria in this game, we will make the following definition (refer to Figure 6).

Definition 5. *For any $(x, y) \in PF(S)$, define $\tilde{x}(x) = \gamma_1((b_1, 0), (x, f(x)))$ and $\tilde{y}(y) = \gamma_2((f^{-1}(y), y), (0, b_2))$.*

[9]Howard (1992) extended the original Nash demand game to a multi-stage game which allows for "renegotiation". In Anbarci and Boyd (2011), their second Nash demand game introduced an arbitration stage, in which the rule of "equally splitting the difference" is utilized to decide the arbitration outcome.

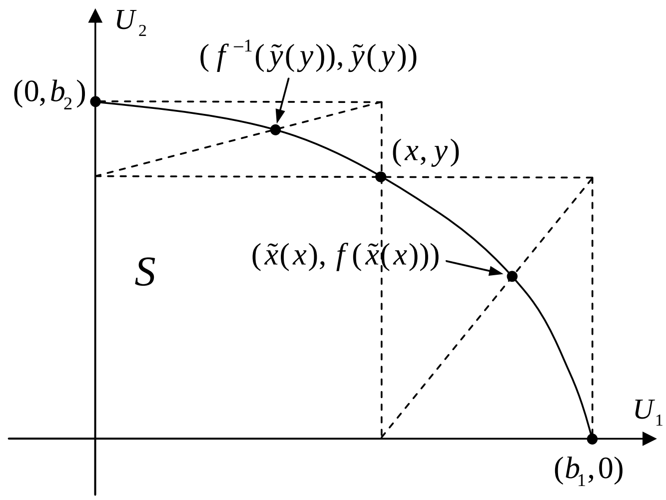

Figure 6: Definition of $\tilde{x}(x)$ and $\tilde{y}(y)$.

$\tilde{x}(x)$ is player 1's stage-2 payoff when his opponent's makes the offer $(x, y) \in PF(S)$ while he makes the extreme offer $(b_1, 0)$. On the other hand, $\tilde{y}(y)$ is player 2's stage-2 payoff when his opponent makes the offer $(x, y) \in PF(S)$ while he makes the extreme offer $(0, b_2)$. According to Lemma 1, a player's arbitrated payoff is strictly increasing in his own demand. Thus, $\tilde{x}(x)$ is player 1's best possible (stage-2) arbitrated payoff when his opponent makes the offer (x, y). Similarly, $\tilde{y}(y)$ is player 2's best possible (stage-2) arbitrated payoff when his opponent makes the offer (x, y).

We will use $((x_1, y_1), (x_2, y_2))$ to denote the strategy profile in which player 1 submits the offer (x_1, y_1) and player 2 submits the offer (x_2, y_2). If a player makes the offer (x, y), then the other player can choose to make the same offer (x, y) and obtain x (if he is player 1) or y (if he is player 2), or choose to make the extreme offer (which will move the game to arbitration) and obtain $\tilde{x}(x)$ (if he is player 1) or $\tilde{y}(y)$ (if he is player 2) at the arbitration stage. Thus, $((x, y), (x, y))$ is a Nash equilibrium if and only if $x \geq \delta\tilde{x}(x)$ and $y \geq \delta\tilde{y}(y)$. In addition, $((b_1, 0), (0, b_2))$ is always a Nash equilibrium regardless of how high the discount factor might be.

The following theorem summarizes the results above. It actually describes all the possible Nash equilibria in the simultaneous-offer game with symmetric arbitration.

Theorem 3. *In the simultaneous-offer game with symmetric arbitration, there are two possible types of Nash equilibria:*

(i) (immediate-agreement equilibrium) $((x, y), (x, y))$ $((x, y) \in PF(S))$ *is a*

Nash equilibrium if and only if $x \geq \delta\tilde{x}(x)$ *and* $y \geq \delta\tilde{y}(y)$*;*

(ii) (arbitration equilibrium) $((b_1, 0), (0, b_2))$ *is a Nash equilibrium for any* $\delta \in (0, 1]$.

Proof: see the appendix. □

As will be illustrated in the following example, both types of Nash equilibria described in Theorem 3 appear as the discount factor changes from 0 to 1. Moreover, for some range of discount factors, the Nash equilibrium is not unique.

Example 1. *Assume that* $b_1 = b_2 = 1$ *and* $f(x) = 1 - x^2$ *for* $x \in [0, 1]$*. Assume that the bargaining game is the simultaneous-offer game with symmetric arbitration.*

Analysis of the example: Table 1 lists the equilibrium type of the game and Figure 7 depicts the equilibrium payoff(s) of player 1. When $0 < \delta \leq 0.741$, there exist multiple Nash equilibria which include both the equilibrium with immediate agreement and the equilibrium with arbitration. Notice that although the equilibrium with arbitration is unique, the equilibrium with immediate agreement is not unique (except at $\delta = 0.741$). The range of player 1's payoffs obtained from equilibria with immediate agreement expands as the discount factor becomes small. As δ approaches zero, this range approaches $[0, 1]$, which means that any point on the Pareto frontier can be supported as the payoff of an immediate-agreement equilibrium.

δ	**Equilibrium Type**
$0 < \delta \leq 0.741$	Immediate-agreement, Arbitration
$0.741 < \delta \leq 1$	Arbitration

Table 1: Nash equilibrium of the game in Example 1.

When $0.741 < \delta \leq 1$, the unique Nash equilibrium is an equilibrium with arbitration. Notice that as δ approaches 1, the equilibrium payoff of player 1 converges to the payoff that he would receive from the Kalai-Smorodinsky (KS) solution outcome (we will further illustrate this point in Theorem 4). □

In Example 1, when the discount factor is large, the unique Nash equilibrium is an equilibrium with arbitration; when the discount factor is small, then besides the equilibrium with arbitration, the equilibrium with immediate

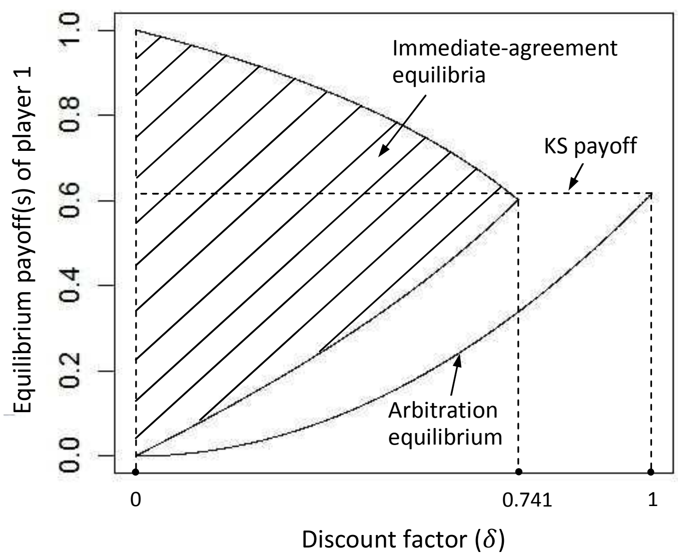

Figure 7: Equilibrium payoff(s) of player 1.

agreement also appears. These two results turn out to be general properties that are true for any bargaining set $S \in \mathcal{B}$. The next theorem (Theorem 4) summarizes these results.

Define $x^*(\delta)$ as the unique $x \in [0, b_1]$ that satisfies $\delta\tilde{x}(x) = x$, and $y^*(\delta)$ as the unique $y \in [0, b_2]$ that satisfies $\delta\tilde{y}(y) = y$. We have:

Theorem 4. *In the simultaneous-offer game with symmetric arbitration, there exists a* $\hat{\delta} \in (0, 1)$*, such that (i) if* $\delta \in (0, \hat{\delta}]$*, then for any* $x \in [x^*(\delta), f^{-1}(y^*(\delta))]$ *(which is nonempty),* $((x, f(x)), (x, f(x)))$ *is a Nash equilibrium;*[10] *and (ii) if* $\delta \in (\hat{\delta}, 1]$*, then* $((b_1, 0), (0, b_2))$ *is the only Nash equilibrium, and the stage 2 arbitrated outcome for the equilibrium* $((b_1, 0), (0, b_2))$*,* $\gamma((b_1, 0), (0, b_2))$*, coincides with the Kalai-Smorodinsky solution outcome of the Nash bargaining problem* $((0, 0), S)$.

Proof: see the appendix. □

According to Theorem 3 (ii), when the discount factor becomes close to 1, the unique equilibrium outcome of the simultaneous-offer game with symmetric arbitration coincides with the Kalai-Smorodinsky solution outcome of the Nash bargaining problem $((0, 0), S)$. To see this, notice that the Kalai-Smorodinsky solution to the Nash bargaining problem $((0, 0), S)$ is the intersection point of $L((0, 0), (b_1, b_2))$ and the Pareto frontier (Kalai and Smorodinsky, 1975). The

[10] In addition, notice that $((b_1, 0), (0, b_2))$ is always a Nash equilibrium.

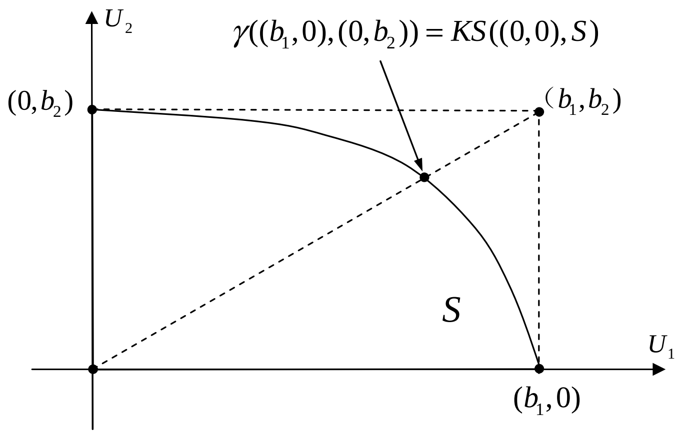

Figure 8: Equilibrium outcome when δ is close to 1.

unique equilibrium outcome of our simultaneous-offer game when the discount factor is close to 1 is $\gamma((b_1, 0), (0, b_2))$. Refer to Figure 8. It is obvious that $\gamma((b_1, 0), (0, b_2)) = KS((0, 0), S)$.

The key axiom that leads to the *symmetric arbitration solution* is the axiom of Symmetry in Offers and the key axiom that leads to the *Kalai-Smorodinsky solution* is the axiom of Individual Monotonicity.[11] Those two axioms have totally different meanings and it is surprising that if we introduce arbitration in the simultaneous-offer game and require the arbitrator to obey the axiom of Symmetry in Offers (and the other two axioms), then the equilibrium outcome of the simultaneous-offer game will be the same as the Kalai-Smorodinsky solution outcome (as soon as δ is close to 1).

Corollary 3 of Anbarci and Boyd (2011) shows that when the continuation probability is *small*, the Kalai-Smorodinsky solution outcome must be one of the equilibrium outcomes. Moreover, the underlying equilibrium is an equilibrium with immediate agreement. Our result shows that when the discount factor is *large*, the Kalai-Smorodinsky solution outcome is the unique equilibrium outcome. Moreover, the underlying equilibrium is an equilibrium with arbitration.

When the bargaining set has a linear Pareto frontier, it can be verified that the threshold discount factor $\hat{\delta}$ in Theorem 4 is $\frac{2}{3}$, regardless of what the slope

[11]The Kalai-Smorodinsky solution is the axiomatic solution that satisfies the following four axioms: Invariance w.r.t. Affine Transformation, Pareto Optimality, Symmetry and Individual Monotonicity. The Kalai-Smorodinsky solution differs from the Nash solution by replacing the axiom of Independence of Irrelevant Alternatives with the axiom of Individual Monotonicity.

of the Pareto frontier might be. This threshold is the same as the threshold continuation probability obtained in Anbarci and Boyd (2011).[12] This is not surprising because (i) the continuation probability in Anbarci and Boyd (2011) is equivalent to the discount factor in our game (see also footnote 2), and (ii) the symmetric arbitration solution coincides with the rule of "equally splitting the difference" when the Pareto frontier is linear.

5.2 Alternating-Offer Game

This subsection will propose and analyze an alternating-offer game that involves symmetric arbitration.[13] In particular, we define the *alternating-offer game with symmetric arbitration* as the following three-stage procedure:

1. Stage 1: player 1 makes an offer $(x_1, y_1) \in PF(S)$ and player 2 decides whether to accept the offer, ending the game with (x_1, y_1), or reject the offer, moving the game on to the next stage;

2. Stage 2: player 2 makes an offer $(x_2, y_2) \in PF(S)$ and player 1 decides whether to accept the offer, ending the game with (x_2, y_2), or reject the offer, moving the game on to the final stage (arbitration stage);

3. Stage 3: an arbitrator decides the final outcome using the *symmetric arbitration solution*, i.e., $\gamma((x_1, y_1), (x_2, y_2))$ is the arbitrated outcome.[14]

Players' payoffs obtained at stage i are subject to a discount of δ^{i-1}. We will characterize the subgame perfect equilibria (henceforth SPE) of this game. We first impose two tie-breaking rules and make some definitions.

Tie-breaking rule 1: whenever a player is indifferent between acceptance and rejection, he always chooses acceptance.

[12]The proof of corollary 3 of Anbarci and Boyd (2011) suggests that when the Pareto frontier is linear, the equilibrium with immediate agreement appears only if the continuation probability is less than $\frac{2}{3}$.

[13]Our game defined below is a variant of the alternating-offer game proposed by Yildiz (2011). Yildiz (2011) assumed that two players make offers sequentially and that if both offers are rejected by opponents, then the final offer arbitration rule is used to decide the final outcome. The final offer arbitration rule used by Yildiz (2011) is such that the offer that yields the higher Nash product is chosen as the arbitration outcome. It turns out that the unique subgame perfect equilibrium outcome in his game coincides with the equilibrium outcome in Rubinstein's infinite-horizon alternating-offer bargaining game (Rubinstein, 1982).

[14]We assume that if (x_1, y_1) and (x_2, y_2) are compatible, then each player gets what he demands at stage 2. Notice that in equilibrium, player 2 will never make an offer that is incompatible with player 1's offer.

Tie-breaking rule 2: whenever a player is indifferent between the two options that he will offer his opponent, he always chooses the option that brings a higher payoff for his opponent.

Definition 6. *For any* $(x_1, y_1) \in PF(S)$ *with* $(x_1, y_1) \neq (0, b_2)$ *and* $(x_2, y_2) \in PF(S)$ *with* $x_2 \leq \delta x_1$*, define the following points (refer to Figure 9):* $A = (x_2, y_2)$*;* $B = (x_2, f(\frac{1}{\delta}x_2))$*;* $C = (\frac{1}{\delta}x_2, f(\frac{1}{\delta}x_2))$*;* $D = (\frac{1}{\delta}x_2, y_1)$ *and* $E = (x_1, y_1)$*.*

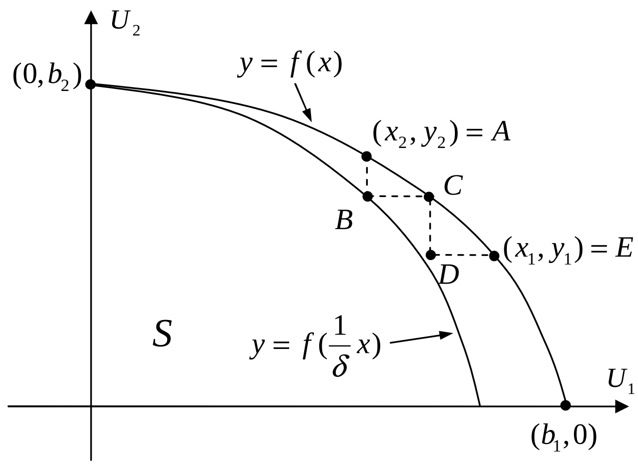

Figure 9: Definitions of points A, B, C, D and E.

The points A, B, C, D and E implicitly depend on (x_1, y_1) and/or (x_2, y_2). However, for simplicity, we omit that dependence in the notation. The curve $y = f(\frac{1}{\delta}x)$ in Figure 9 is obtained by fixing the payoff of player 2 and scaling down the payoff of player 1 by the discount factor δ.[15] Thus, for player 1, he must be indifferent between accepting the outcome B at the current stage and accepting the outcome C at the next stage. It should be noted that the point D is typically not on the curve $y = f(\frac{1}{\delta}x)$.

Definition 7. *For any given* $(x_1, y_1) \in PF(S)$ *with* $(x_1, y_1) \neq (0, b_2)$*, define* $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$ *as the unique point* $(x_2, y_2) \in PF(S)$ *that satisfies: (i)* $x_2 \leq \delta x_1$*; (ii)* $|AB| * |BC| = |CD| * |DE|$*.*

The point $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$ is well-defined because as $(x_2, y_2) \in PF(S)$ moves along the Pareto frontier from $(0, b_2)$ to $(\delta x_1, f(\delta x_1))$, $|AB| * |BC|$ strictly increases from zero to some positive number and $|CD| * |DE|$ strictly decreases from a positive number to zero. If $|AB| * |BC| = |CD| * |DE|$,

[15] To see this, note $y = f(\frac{1}{\delta}x)$ can be rewritten as $x = \delta f^{-1}(y)$.

then the main diagonal of the rectangle $AMEN$ must intersect the Pareto frontier at the point C (see Figure 10). That is, we must have $C = \gamma((x_1, y_1), (\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1)))$. Since for player 1, $\delta C \sim B$, we thus have $\delta\gamma_1((x_1, y_1), (\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))) = \hat{x}_2(x_1, y_1)$. The following lemma further shows that for any given (x_1, y_1), the point $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$ is actually the only point on the Pareto frontier that satisfies $\delta\gamma_1((x_1, y_1), (x_2, y_2)) = x_2$. It also shows that $\hat{x}_2(x_1, y_1)$ is strictly increasing in x_1.

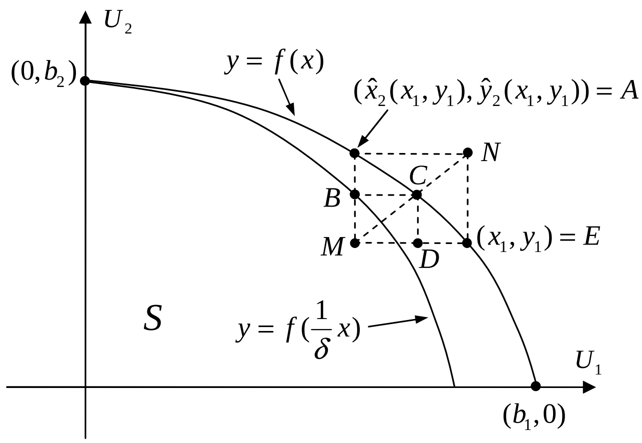

Figure 10: Definition of $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$.

Lemma 2. *For* $(x_1, y_1) \in PF(S)$ *with* $(x_1, y_1) \neq (0, b_2)$, *we have:*

(i) $\delta\gamma_1((x_1, y_1), (\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))) = \hat{x}_2(x_1, y_1)$;

(ii) for any $(x_2, y_2) \in PF(S)$ *with* $x_2 < \hat{x}_2(x_1, y_1)$, *we have:* $\delta\gamma_1((x_1, y_1), (x_2, y_2)) > x_2$;

(iii) for any $(x_2, y_2) \in PF(S)$ *with* $x_2 > \hat{x}_2(x_1, y_1)$, *we have:* $\delta\gamma_1((x_1, y_1), (x_2, y_2)) < x_2$;

(iv) if $(x_1', y_1') \in PF(S)$ *and* $x_1' > x_1$, *then we have:* $\hat{x}_2(x_1', y_1') > \hat{x}_2(x_1, y_1)$.

Proof: see the appendix. □

An implication of Lemma 2 (i) (ii) and (iii) is that, if the game were at stage 2 and player 2 made the offer (x_2, y_2), then whether or not player 1 accepts the offer depends on whether or not x_2 is greater than $\hat{x}_2(x_1, y_1)$. That is, we have:

Corollary 1. *Suppose player 1 offers* $(x_1, y_1) \neq (0, b_2)$ *at stage 1 which player 2 rejects and player 2 makes an offer* (x_2, y_2) *at stage 2, then player 1 will accept the offer* (x_2, y_2) *if and only if* $x_2 \geq \hat{x}_2(x_1, y_1)$.

The following lemma characterizes the players' equilibrium behavior at stage 2. It is essential for our main result in characterizing the SPE of our entire game.

Lemma 3. *In equilibrium, if at stage 1, player 1 offers* $(x_1, y_1) \neq (0, b_2)$ *which player 2 rejects, then at stage 2, we have:*

(i) player 2 must either offer $(0, b_2)$ *which player 1 rejects, or offer* $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$ *which player 1 accepts.*

(ii) if $(x_1, y_1) \neq (b_1, 0)$*, then player 2 must be indifferent between offering* $(0, b_2)$ *and offering* $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$*, i.e.,* $\delta\gamma_2((x_1, y_1), (0, b_2)) = \hat{y}_2(x_1, y_1)$*.*

Proof: see the appendix. □

The intuition of Lemma 3 (i) is straightforward. If player 1 offers $(x_1, y_1) \neq (0, b_2)$ which player 2 rejects, then at stage 2, player 2 can either make an offer that player 1 will reject or make an offer that player 1 will accept. In the former case, player 2's best option is to make the extreme offer $(0, b_2)$, because the more demand he makes in his offer, the more arbitrated payoff he can obtain at the arbitration stage (according to Lemma 1). In the latter case, player 2's best option is to make the offer $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$, because his offer (x_2, y_2) will be accepted by player 1 if and only if $x_2 \geq \hat{x}_2(x_1, y_1)$ (according to Corollary 1).

Lemma 3 (ii) states that as soon as $(x_1, y_1) \notin \{(0, b_2), (b_1, 0)\}$ is rejected by player 2 at stage 1, then player 2 must be indifferent between making the extreme offer (i.e., offering $(0, b_2)$) and "concession" (i.e., offering $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$) at stage 2. This is because, if player 2 strictly prefers one option over the other, say, player 2 strictly prefers offering $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$ over offering $(0, b_2)$, then at stage 1, player 1 has the incentive to deviate to a slightly more extreme offer $(x_1 + \epsilon, f(x_1 + \epsilon))$. Such a small deviation will not change player 2's preference over the two options at stage 2, i.e., player 2 strictly prefers offering $(\hat{x}_2(x_1 + \epsilon, f(x_1 + \epsilon)), \hat{y}_2(x_1 + \epsilon, f(x_1 + \epsilon)))$ over offering $(0, b_2)$. As a result, after deviation, player 1 obtains a payoff of $\hat{x}_2(x_1 + \epsilon, f(x_1 + \epsilon))$ which is higher than $\hat{x}_2(x_1, y_1)$, the payoff before deviation (according to Lemma 2 (iv)).

A direct result of Lemma 3 (i) is the following theorem, which characterizes all SPE of the game. Note that in equilibrium, player 1 will never offer $(0, b_2)$ at stage 1 because the offer $(0, b_2)$ is dominated by the offer $(b_1, 0)$ which will bring him a payoff of at least $\delta^2\gamma((b_1, 0), (0, b_2)) > 0$. In addition, using tie-breaking rule 1 and tie-breaking rule 2, it can be shown that the SPE of the game must be unique.

Theorem 5. *In the alternating-offer game with symmetric arbitration, there exists a unique SPE and the unique SPE must take one of the following three forms:*

(i) (immediate-agreement) at stage 1, player 1 offers $(x_1, y_1) \neq (0, b_2)$ *which player 2 accepts;*

(ii) (delayed-agreement) at stage 1, player 1 offers $(x_1, y_1) \neq (0, b_2)$ *which player 2 rejects; at stage 2, player 2 offers* $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$ *which player 1 accepts;*

(iii) (arbitration) at stage 1, player 1 offers $(x_1, y_1) \neq (0, b_2)$ *which player 2 rejects; at stage 2, player 2 offers* $(0, b_2)$ *which player 1 rejects.*

Theorem 5 states that the unique SPE of the alternating-offer game with symmetric arbitration is either an equilibrium with immediate agreement, or an equilibrium with delayed agreement, or an equilibrium with arbitration. The equilibrium type depends on the discount factor. This dependency may be very complex. However, the next theorem shows that as long as the discount factor is sufficiently large, then the unique SPE of the game must be an equilibrium with arbitration and the corresponding arbitration outcome coincides with the Kalai-Smorodinsky solution outcome.

Theorem 6. *There exists a* $\delta^* \in (0, 1)$ *with* $0 < \delta^* < 1$*, such that when* $\delta \in (\delta^*, 1]$*, the unique SPE of the alternating-offer game with symmetric arbitration is that at stage 1, player 1 makes the offer* $(b_1, 0)$ *which player 2 rejects, and at stage 2, player 2 makes the offer* $(0, b_2)$ *which player 1 rejects; the equilibrium outcome of the game is thus* $\gamma((b_1, 0), (0, b_2))$ *which coincides with the Kalai-Smorodinsky solution outcome of the Nash bargaining problem* $((0, 0), S)$*.*

Proof: see the appendix. □

When the Pareto frontier of the bargaining set is linear, it can be verified that the threshold discount factor δ^* is 0.91. This threshold is much larger than the threshold discount factor obtained in the simultaneous-offer game for the linear Pareto frontier case. This is because there are three stages in the alternating-offer game, but only two stages in the simultaneous-offer game. In other words, for a given discount factor, the arbitration outcome is discounted more severely in the alternating-offer game. As a result, in the alternating-offer game, players have less incentive to make extreme offers and the result that players make extreme offers in equilibrium is less robust.

6 Conclusion

This paper defines a class of arbitration problems and characterizes its solution using the axiomatic approach. We impose three axioms that an arbitrator should use. They are "Symmetry in Offers", "Invariance" and "Pareto Optimality". The key rule, Symmetry in Offers, requires that whenever players' offers are symmetric, the arbitrated outcome should also be symmetric. We show that there is a unique arbitration solution, called the symmetric arbitration solution, that satisfies all three axioms. The symmetric arbitration solution has a simple graphical representation.

We then introduce symmetric arbitration in two bargaining games. One is the simultaneous-offer game and the other is the alternating-offer game. At the arbitration stage of both games, the arbitrator uses the symmetric arbitration solution to decide the arbitration outcome. We show that in both games, if the discount factor is sufficiently close to 1, then the *unique* equilibrium is such that both players make extreme offers and the corresponding equilibrium outcome is the Kalai-Smorodinsky solution outcome.

Although the equilibrium outcomes of the two games coincide with that of the Kalai-Smorodinsky solution (when δ is close to 1), our result is not a typical implementation result. Strictly speaking, a strategic implementation of an axiomatic bargaining solution requires that the mechanism used for implementation can be translated into a form that only depends on the physical outcomes of bargaining and not on the players' preferences or utility representations (Serrano, 1997; Dagan and Serrano, 1998). Our games cannot be translated into a form that only depends on the physical outcomes, because the symmetric arbitration solution is defined on the basis of players' utilities. However, compared with the implementation mechanism on the Kalai-Smorodinsky solution in the literature[16], our games are much more simple. Our games also help us to understand the Kalai-Smorodinsky solution from a new perspective. That is, the Kalai-Smorodinsky solution outcome is the only fair and efficient arbitration outcome when both players make extreme offers.

Although our model assumes that both players have the same discount factor, our main result can be extended to the case where the two players' discount factors differ. That is, as long as the discount factors of the two players are sufficiently close to 1, then the unique equilibrium outcome of both the simultaneous-offer game and the alternating-offer game coincides with the

[16]See, for example, the "auctioning fractions of dictatorship" mechanism proposed by Moulin (1984).

Kalai-Smorodinsky solution outcome.

Appendix

Proof of Lemma 1

Let's consider γ_1. For any given $x_1 \in [d_1, b_1]$ and $x'_1 \in [d_1, b_1]$ with $x_1 < x'_1$ and any $x_2 \in [d_2, b_2]$ with $x_2 < x_1$, the line connecting $(x_1, f(x_1)) \wedge (x_2, f(x_2))$ and $(x_1, f(x_1)) \vee (x_2, f(x_2))$ is strictly above the line connecting $(x'_1, f(x'_1)) \wedge (x_2, f(x_2))$ and $(x'_1, f(x'_1)) \vee (x_2, f(x_2))$ (see Figure 11). Since the Pareto frontier is strictly downward-sloping, we must have $\gamma_1((x_1, f(x_1)), (x_2, f(x_2))) < \gamma_1((x'_1, f(x'_1)), (x_2, f(x_2)))$. Thus, $\gamma_1((x_1, f(x_1)), (x_2, f(x_2)))$ is strictly increasing in x_1. Similarly, $\gamma_1((x_1, f(x_1)), (x_2, f(x_2)))$ is strictly increasing in x_2.

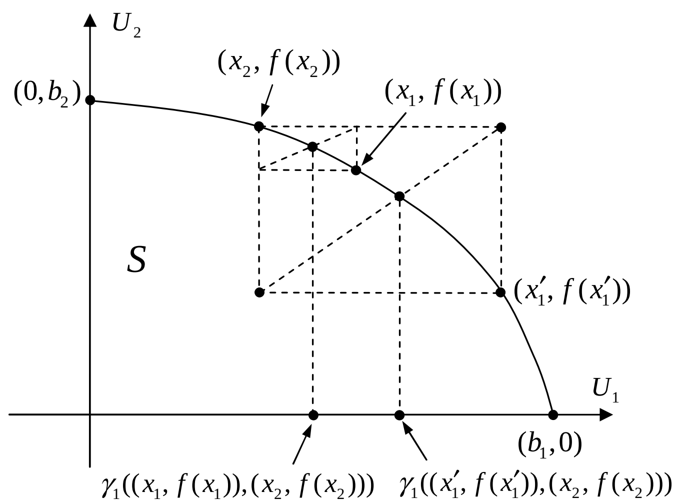

Figure 11: $\gamma_1((x_1, f(x_1)), (x_2, f(x_2)))$ and $\gamma_1((x'_1, f(x'_1)), (x_2, f(x_2)))$ where $x_1 < x'_1$.

The proof for γ_2 is similar and is omitted.

Proof of Theorem 3

(i) The result holds if the following is true:

(a) if player 2's offer is $(x, y) \in PF(S)$, then (x, y) is player 1's best response if and only if $x \geq \delta\tilde{x}(x)$; (b) if player 1's offer is $(x, y) \in PF(S)$, then (x, y) is player 2's best response if and only if $y \geq \delta\tilde{y}(y)$.

We will only prove (a) in the following. The proof of (b) is similar.

Suppose player 2's offer is (x, y). If player 1 makes an offer $(x', y') \in PF(S)$ with $0 \leq x' < x$, then since (x', y') and (x, y) are compatible, player 1's payoff must be x', which is strictly less than x. If player 1 makes an offer $(x', y') \in PF(S)$ with $x < x' \leq b_1$, then by Lemma 1, his payoff is at most $\delta\tilde{x}(x)$. Thus, we have shown that (x, y) is player 1's best response if and only if $x \geq \delta\tilde{x}(x)$.

(ii) We will show that for any $\delta \in (0, 1]$, $((b_1, 0), (0, b_2))$ is a Nash equilibrium. We will first show that $(b_1, 0)$ is player 1's best response to player 2's offer $(0, b_2)$. Suppose player 2's offer is $(0, b_2)$, then player 1 can either make the offer $(0, b_2)$ or make some offer $(x, f(x)) \neq (0, b_2)$. If player 1 offers $(0, b_2)$, then his payoff is 0. If player 1 offers $(x, f(x)) \neq (0, b_2)$, then the game will move to the arbitration stage and player 1's payoff is $\delta\gamma_1((x, f(x)), (0, b_2)) > 0$. Now, by Lemma 1, $\delta\gamma_1((x, y), (0, b_2))$ is strictly increasing in x, so player 1's best response to player 2's offer $(0, b_2)$ is $(b_1, 0)$. Similarly, player 2's best response to player 1's offer $(b_1, 0)$ is $(0, b_2)$. Thus, $((b_1, 0), (0, b_2))$ is a Nash equilibrium for any $\delta \in (0, 1]$.

At last, note that $((x, y), (x, y))$ $((x, y) \in PF(S))$ and $((b_1, 0), (0, b_2))$ are the only two possible types of Nash equilibria, i.e., any $((x_1, f(x_1)), (x_2, f(x_2)))$ with $((x_1, f(x_1)), (x_2, f(x_2))) \neq ((b_1, 0), (0, b_2))$ and $x_1 \neq x_2$ cannot be the Nash equilibrium for any $\delta \in (0, 1]$. This is because (i) if $x_1 < x_2$, then the two offers are compatible and player 1 has incentive to deviate to $(x_1 + \epsilon, f(x_1 + \epsilon))$ with some $x_1 + \epsilon \leq x_2$; (ii) if $x_1 > x_2$, then by Lemma 1, the player who does not make the extreme offer has the incentive to deviate to making the extreme offer.

Proof of Theorem 4

First notice that $x^*(\delta)$ is well-defined because (i) $\delta\tilde{x}(x) - x = (\tilde{x}(x) - x) - (1 - \delta)\tilde{x}(x)$ is strictly decreasing in x (see Figure 12), (ii) $\delta\tilde{x}(x) - x > 0$ at $x = 0$, and (iii) $\delta\tilde{x}(x) - x \leq 0$ at $x = b_1$. Similarly, $y^*(\delta)$ is well-defined.

Since $\delta\tilde{x}(x) - x$ is strictly decreasing in x, $x \geq \delta\tilde{x}(x)$ if and only if $x \geq x^*(\delta)$. That is, if player 2's offer is $(x, y) \in PF(S)$, then player 1's best response is to make the same offer if and only if $x \geq x^*(\delta)$. Similarly, if player 1's offer is $(x, y) \in PF(S)$, then player 2's best response is to make the same offer if and only if $y \geq y^*(\delta)$ (see Figure 13).

Observing that $x^*(\delta) \to b_1$ and $f^{-1}(y^*(\delta)) \to 0$ as $\delta \to 1$, and $x^*(\delta) \to 0$ and $f^{-1}(y^*(\delta)) \to b_1$ as $\delta \to 0$, there exists a unique $\delta \in (0, 1)$, denoted by $\hat{\delta}$, that satisfies $x^*(\delta) = f^{-1}(y^*(\delta))$.

According to Lemma 3, $((x, y), (x, y))$ $((x, y) \in PF(S))$ is a Nash equilibrium if and only if $x \geq \delta\tilde{x}(x)$ and $y \geq \delta\tilde{y}(y)$. So, $((x, y), (x, y))$ $((x, y) \in$

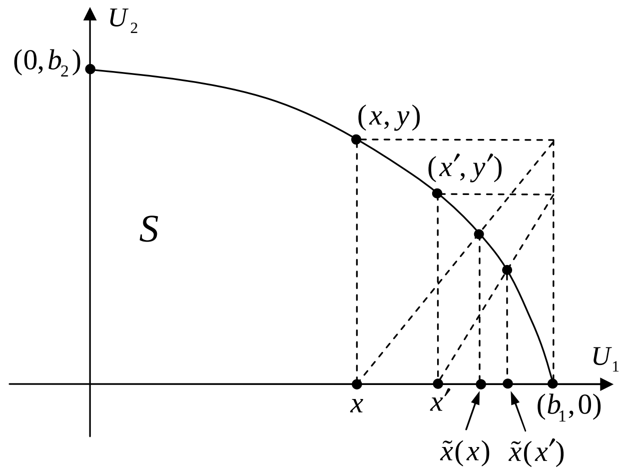

Figure 12: $\tilde{x}(x) - x$ is strictly decreasing in x and $\tilde{x}(x)$ is strictly increasing in x.

$PF(S))$ is a Nash equilibrium if and only if $x^*(\delta) \leq x \leq f^{-1}(y^*(\delta))$. The remainder of the proof is straightforward and is omitted.

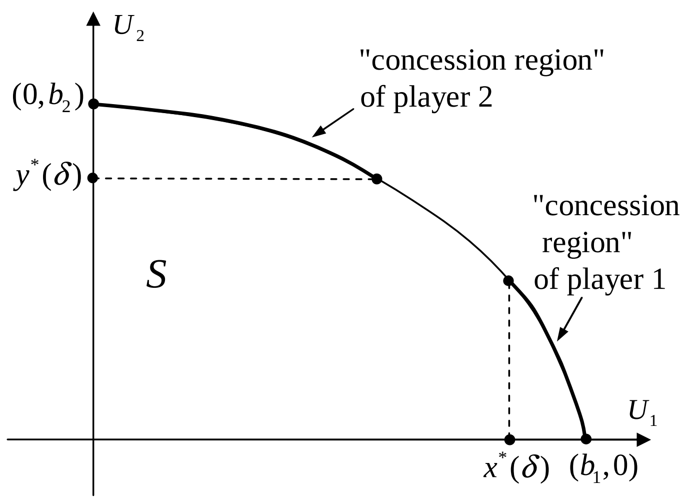

Figure 13: The regions that players will choose "concession" instead of making the extreme offers.

Proof of Lemma 2

(i) Refer to Figure 10. By definition, for any given (x_1, y_1) with $(x_1, y_1) \neq (0, b_2)$, the pair $((x_1, y_1), (\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1)))$ satisfies $|AB| * |BC| = |CD| *$

$|DE|$. This implies that the point C must be on the line that connects $(x_1, y_1) \wedge (x_2, y_2)$ (i.e., M) and $(x_1, y_1) \vee (x_2, y_2)$ (i.e., N). In addition, notice that point C is on $PF(S)$. Then, we must have $C = \gamma((x_1, y_1), (\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1)))$. Since by definition, $C = (\frac{1}{\delta}\hat{x}_2(x_1, y_1), f(\frac{1}{\delta}\hat{x}_2(x_1, y_1)))$, we have:

$$\delta\gamma_1((x_1, y_1), (\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))) = \hat{x}_2(x_1, y_1).$$

(ii) Refer to Figure 14. Note that for given (x_1, y_1) with $(x_1, y_1) \neq (0, b_2)$, as (x_2, y_2) moves from the lower-right to the upper-left along the Pareto frontier, the corresponding $|AB| * |BC|$ strictly decreases and $|CD| * |DE|$ strictly increases. Thus, for $(x_2, y_2) \in PF(S)$ with $x_2 < \hat{x}_2(x_1, y_1)$, we must have $|AB| * |BC| < |CD| * |DE|$. This implies the slope of the line MC is bigger than that of the line MN. Thus, the point O (the intersection point of the line MN with $PF(S)$) must be on the right of the line CD, then we have: $\gamma_1((x_1, y_1), (x_2, y_2)) > \frac{1}{\delta}x_2$, i.e., $\delta\gamma_1((x_1, y_1), (x_2, y_2)) > x_2$.

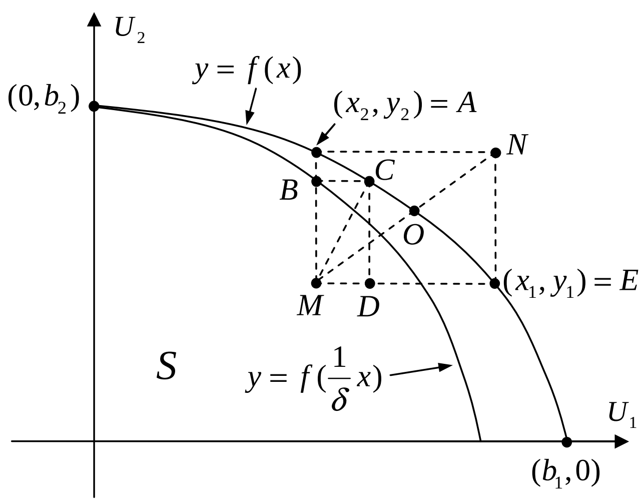

Figure 14: The case where $x_2 < \hat{x}(x_1, y_1)$.

(iii) We have three sub-cases here:

(a) $\hat{x}_2(x_1, y_1) < x_2 \leq \delta x_1$

Refer to Figure 15. Note that for any given (x_1, y_1) with $(x_1, y_1) \neq (0, b_2)$, as (x_2, y_2) moves from the upper-left to the lower-right along the Pareto frontier, the corresponding $|AB| * |BC|$ strictly increases and $|CD| * |DE|$ strictly decreases. Thus, for $(x_2, y_2) \in PF(S)$ with $x_2 > \hat{x}_2(x_1, y_1)$, we must have $|AB| * |BC| > |CD| * |DE|$. This implies the slope of the line MC is smaller than that of the line MN. Thus, the point O must be on the left of the line CD, then we have: $\gamma_1((x_1, y_1), (x_2, y_2)) < \frac{1}{\delta}x_2$, i.e., $\delta\gamma_1((x_1, y_1), (x_2, y_2)) < x_2$.

(b) $\delta x_1 < x_2 \leq x_1$

For this case, since $x_2 \leq x_1$, then we must have $\gamma_1((x_1, y_1), (x_2, y_2)) \leq x_1$. Then: $\delta\gamma_1((x_1, y_1), (x_2, y_2)) \leq \delta x_1 < x_2$.

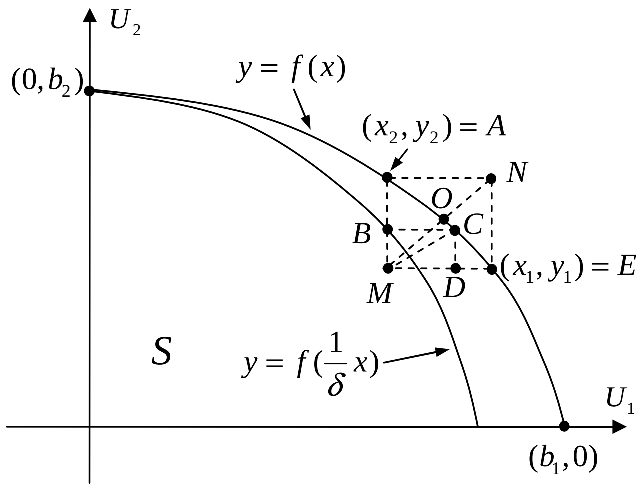

Figure 15: The case where $\hat{x}_2(x_1, y_1) < x_2 \leq \delta x_1$.

(c) $x_2 > x_1$

For this case, since $x_2 > x_1$, then we have $\gamma_1((x_1, y_1), (x_2, y_2)) < x_2$. Thus, $\delta\gamma_1((x_1, y_1), (x_2, y_2)) < x_2$.

(iv) Refer to Figure 16. Suppose we have $(x_1', y_1') \in PF(S)$ and $x_1' > x_1$. Now, for (x_1, y_1) and $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$, we have:

$$|AB| * |BC| = |CD| * |DE|$$

Since (x_1', y_1') is on the lower right of (x, y), we have:

$$|AB| * |BC| < |CD'| * |D'E'|.$$

Again, note that for given (x_1, y_1), as (x_2, y_2) moves from the upper-left to the lower-right along the Pareto frontier, $|AB| * |BC|$ strictly increases, and $|BC| * |CD|$ strictly decreases. So, we must have

$$\hat{x}_2(x_1', y_1') > \hat{x}_2(x_1, y_1).$$

Proof of Lemma 3

(i) The proof is obvious and is omitted.

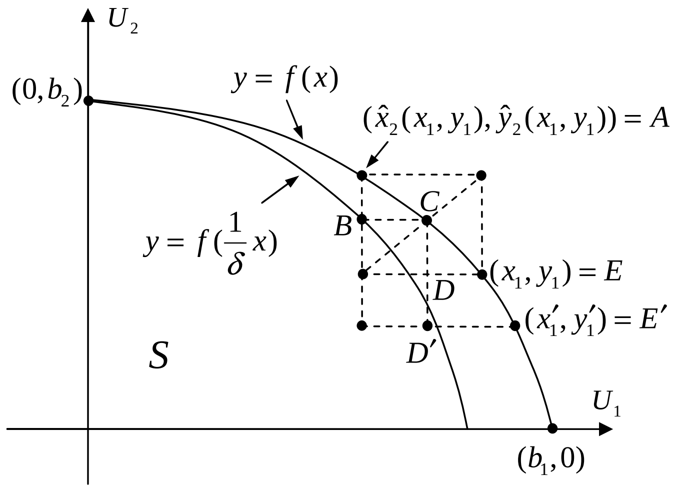

Figure 16: A comparison of $\hat{x}_2(x_1', y_1')$ and $\hat{x}_2(x_1, y_1)$ where $x_1' > x_1$.

(ii) Suppose player 1 offers $(x_1, y_1) \notin \{(0, b_2), (b_1, 0)\}$. By part (i), we know that if player 2 rejects $(x_1, y_1) \neq (0, b_2)$, then at stage 2, he must either offer $(0, b_2)$ or offer $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$. The corresponding (stage 1) payoff for player 2 is either $\delta^2\gamma_2((x_1, y_1), (0, b_2))$ or $\delta\hat{y}_2(x_1, y_1)$. Since player 2 chooses to reject (x_1, y_1) at stage 1, then we must have:

$$max\{\delta^2\gamma_2((x_1, y_1), (0, b_2)), \delta\hat{y}_2(x_1, y_1)\} > y_1.$$

Note that the above inequality is strict because we have assumed that whenever a player is indifferent between "accept" and "reject", he must choose "accept". Now, let's consider the following two cases:

(a) $\delta^2\gamma_2((x_1, y_1), (0, b_2)) > \delta\hat{y}_2(x_1, y_1)$

In this case, player 2 must offer $(0, b_2)$ at stage 2. Player 1 thus obtains a payoff of $\delta^2\gamma_1((x_1, y_1), (0, b_2))$. We will show that player 1 will gain more if he submits a more extreme offer at stage 1. In particular, since $(x_1, y_1) \neq (b_1, 0)$, we can find an $\epsilon' > 0$ such that $x_1' = x_1 + \epsilon' < b_1$, $y_1' = f(x_1')$, $\delta^2\gamma_2((x_1', y_1'), (0, b_2)) > \delta\hat{y}_2(x_1', y_1')$ and $max\{\delta^2\gamma_2((x_1', y_1'), (0, b_2)), \delta\hat{y}_2(x_1', y_1')\} > y_1'$. That is, if player 1 offers (x_1', y_1') at stage 1, then player 2 must reject it and still offer $(0, b_2)$ at stage 2. Thus, player 1 will obtain a payoff of $\delta^2\gamma_1((x_1', y_1'), (0, b_2))$ by offering (x_1', y_1') at stage 1. Now, since $x_1' > x_1$, we have $\delta^2\gamma_1((x_1', y_1'), (0, b_2)) > \delta^2\gamma_1((x_1, y_1), (0, b_2))$. That is, player 1 is better off by offering (x_1', y_1').

(b) $\delta\hat{y}_2(x_1, y_1) > \delta^2\gamma_2((x_1, y_1), (0, b_2))$

In this case, player 2 must offer $(\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))$ at stage 2. Player

1 obtains a payoff of $\delta\hat{x}_2(x_1, y_1)$. Again, we will show that player 1 can gain more if he makes a more extreme offer at stage 1. In particular, since $(x_1, y_1) \neq (b_1, 0)$, we can find an $\epsilon'' > 0$ such that $x_1'' = x_1 + \epsilon'' < b_1$, $y_1'' = f(x_1'')$, $\delta\hat{y}_2(x_1'', y_1'') > \delta^2\gamma_2((x_1'', y_1''), (0, b_2))$ and $max\{\delta^2\gamma_2((x_1'', y_1''), (0, b_2)), \delta\hat{y}_2(x_1'', y_1'')\} > y_1''$. That is, if player 1 offers (x_1'', y_1'') at stage 1, then player 2 must reject it and offer $(\hat{x}_2(x_1'', y_1''), \hat{y}_2(x_1'', y_1''))$ at stage 2. Note $(\hat{x}_2(x_1'', y_1''), \hat{y}_2(x_1'', y_1''))$ must be accepted by player 1. Thus, player 1 will obtain a payoff of $\delta\hat{x}_2(x_1'', y_1'')$ by offering (x_1'', y_1'') at stage 1. Now, by Lemma 2 (iv), since $x_1'' > x_1$, we have: $\delta\hat{x}_2(x_1'', y_1'') > \delta\hat{x}_2(x_1, y_1)$. That is, player 1 is better off by offering (x_1'', y_1'').

Thus, we have proved that if $(x_1, y_1) \notin \{(0, b_2), (b_1, 0)\}$, then we must have $\delta\gamma_2((x_1, y_1), (0, b_2)) = \hat{y}_2(x_1, y_1)$ in equilibrium.

Proof of Theorem 6

We denote $\gamma((b_1, 0), (0, b_2))$ as (x^*, y^*) for simplicity.

Let δ_1^* be the unique $\delta \in (0, 1)$ that satisfies $\delta^2 = \dfrac{2f(\delta^2 x^*)}{f(\delta^2 x^*) + b_2}$. Let $\delta_2^* = \max_{\frac{4}{9}x^* \leq x_1 \leq b_1} \delta_2^*(x_1)$ where $\delta_2^*(x_1)$ is the unique $\delta \in (0, 1)$ that satisfies $\delta\dfrac{f(x_1) + b_2}{2} = f(\dfrac{\frac{\delta}{2}x_1}{1 - \frac{\delta}{2}})$. Let $\delta^* = max\{\delta_1^*, \delta_2^*, \frac{2}{3}\}$. Note since $\delta_1^* \in (0, 1)$ and $\delta_2^* \in (0, 1)$, we have $\delta^* \in (0, 1)$.

The proof is divided into two steps.

First step: We will show that, if $\delta \in (\delta^*, 1]$, then player 1 must offer $(b_1, 0)$ at stage 1.

First, we will show that, if $\delta \in (\delta^*, 1]$, then player 1 will never offer (x_1, y_1) with $x_1 \in [0, \delta^2 x^*)$. In particular, We will show that for player 1, any offer (x_1, y_1) with $x_1 \in [0, \delta^2 x^*)$ is strictly dominated by the offer $(b_1, 0)$.

Note that if player 1 makes the offer (x_1, y_1) with $x_1 \in [0, \delta^2 x^*)$, then his payoff is at most x_1. If player 1 proposes $(b_1, 0)$, then by Lemma 3, player 2 may choose to accept, or reject with counteroffer $(0, b_2)$ which player 1 rejects, or reject with counteroffer $(\hat{x}_2(b_1, 0), \hat{y}_2(b_1, 0))$ which player 1 accepts. If player 2 accepts, then player 1's payoff is b_1; if player 2 rejects with counteroffer $(0, b_2)$ which player 1 rejects, then player 1's payoff is $\delta^2\gamma_1((b_1, 0), (0, b_2)) = \delta^2 x^*$; if player 2 rejects with counteroffer $(\hat{x}_2(b_1, 0), \hat{y}_2(b_1, 0))$ which player 1 will accept, then player 1's payoff is $\delta\hat{x}_2(b_1, 0) = \delta^2\gamma_1((b_1, 0), (\hat{x}_2(b_1, 0), \hat{y}_2(b_1, 0)))$ (the equality is by Lemma 2 (i)). Since $\gamma_1((b_1, 0), (\hat{x}_2(b_1, 0), \hat{y}_2(b_1, 0))) \geq \gamma_1((b_1, 0), (0, b_2))$, we have: $\delta\hat{x}_2(b_1, 0) \geq \delta^2\gamma_1((b_1, 0), (0, b_2)) = \delta^2 x^*$. Thus, we have shown that, if player 1 proposes $(b_1, 0)$ at stage 1, then his payoff is at least $\delta^2 x^*$. Hence, player 1 will never offer (x_1, y_1) with $x_1 \in [0, \delta^2 x^*)$.

Second, we will show that, if $\delta \in (\delta^*, 1]$, then player 1 will never offer (x_1, y_1) with $x_1 \in [\delta^2 x^*, b_1)$. We have the following two cases:

(i) (x_1, y_1) is accepted by player 2.

Note that, if player 2 rejects (x_1, y_1), then his payoff is at least $\delta^2\gamma_2((x_1, y_1), (0, b_2))$. Since player 2 chooses to accept (x_1, y_1), then we must have:

$$\delta^2\gamma_2((x_1, y_1), (0, b_2)) \leq y_1.$$

Since $\delta^2\gamma_2((x_1, y_1), (0, b_2)) \geq \frac{\delta^2}{2}(y_1+b_2)$ (using the fact that the Pareto frontier is strictly "bowed-out"), then we have $\frac{\delta^2}{2}(y_1 + b_2) \leq y_1$, i.e., $\delta^2 \leq \dfrac{2y_1}{y_1 + b_2}$. Since $y_1 = f(x_1) \leq f(\delta^2 x^*) \leq f(\delta_1^{*2} x^*)$ and $\dfrac{2y_1}{y_1 + b_2}$ is increasing in y_1, we have $\delta^2 \leq \dfrac{2f(\delta_1^{*2}x^*)}{f(\delta_1^{*2}x^*) + b_2} = \delta_1^{*2} \leq \delta^{*2}$. Contradiction with $\delta > \delta^*$.

(ii) (x_1, y_1) is rejected by player 2.

We will compare $\delta\gamma_1((x_1, y_1), (0, b_2))$ and $\hat{y}_2(x_1, y_1)$.

First, note that by Lemma 2 (i) and the fact that the Pareto frontier is strictly "bowed-out", we have: $\hat{x}_2(x_1, y_1) = \delta\gamma_1((x_1, y_1), (\hat{x}_2(x_1, y_1), \hat{y}_2(x_1, y_1))) \geq \delta\dfrac{\hat{x}_2(x_1, y_1) + x_1}{2}$. Then we have: $\hat{x}_2(x_1, y_1) \geq \dfrac{\frac{\delta}{2}x_1}{1 - \frac{\delta}{2}}$. Then, $\hat{y}_2(x_1, y_1) \leq f(\dfrac{\frac{\delta}{2}x_1}{1 - \frac{\delta}{2}})$. Therefore,

$$\hat{y}_2(x_1, y_1) \leq f(\frac{\frac{\delta}{2}x_1}{1 - \frac{\delta}{2}}) < \delta\frac{y_1 + b_2}{2} \qquad (5)$$

The last inequality is because $\delta > \delta^* \geq \delta_2^* \geq \delta_2^*(x_1)$,[17] $\delta_2^*(x_1)$ satisfies $f(\dfrac{\frac{\delta_2^*(x_1)}{2}x_1}{1 - \frac{\delta_2^*(x_1)}{2}}) = \delta_2^*(x_1)\dfrac{f(x_1) + b_2}{2}$, $f(\dfrac{\frac{\delta}{2}x_1}{1 - \frac{\delta}{2}})$ is strictly decreasing in δ, $\delta\dfrac{y_1 + b_2}{2}$ is strictly increasing in δ and $y_1 = f(x_1)$.

Now, note $\delta\gamma_2((x_1, y_1), (0, b_2)) \geq \delta\dfrac{y_1 + b_2}{2}$. Then, we have:

$$\delta\gamma_2((x_1, y_1), (0, b_2)) > \hat{y}_2(x_1, y_1).$$

However, by Lemma 3 (ii), if at stage 1, player 1's makes the offer $(x_1, y_1) \notin$

[17] The inequality $\delta_2^* \geq \delta_2^*(x_1)$ is true because of the following. Note that $\delta > \delta^*$ implies $\delta > 2/3$, which implies $\delta_2^* = \max_{\frac{4}{9}x^* \leq x_1 \leq b_1} \delta_2^*(x_1) \geq \max_{\delta^2 x^* \leq x_1 \leq b_1} \delta_2^*(x_1)$. So, for any $x_1 \in [\delta^2 x^*, b_1)$, we have $\delta_2^* \geq \delta_2^*(x_1)$.

$\{(0, b_2), (b_1, 0)\}$ and player 2 rejects it, then we must have

$$\delta \gamma_2((x_1, y_1), (0, b_2)) = \hat{y}_2(x_1, y_1).$$

Contradiction!

Thus, we have proved that player 1 will never offer (x_1, y_1) with $x_1 \in [\delta x^*, 1)$. In addition, we have already proved that player 1 will never offer (x_1, y_1) with $x_1 \in [0, \delta x^*)$. Thus, player 1 must offer $(b_1, 0)$ at Stage 1.

Second Step: We will show that if $\delta \in (\delta^*, 1]$, and if player 1 offers $(b_1, 0)$ at stage 1, then player 1 must reject it and offers $(0, b_2)$ at stage 2.

If $(b_1, 0)$ is proposed by player 1 at stage 1, then by Lemma 2, player 2 has three options: (a) accept $(b_1, 0)$ – player 2's payoff is 0; (b) reject $(b_1, 0)$ and makes the counteroffer $(0, b_2)$ – player 2's payoff is $\delta^2 \gamma_2((b_1, 0), (0, b_2))$; and (c) reject $(b_1, 0)$ and makes the counteroffer $(\hat{x}_2(b_1, 0), \hat{y}_2(b_1, 0))$ – player 2's payoff is $\delta \hat{y}_2(b_1, 0)$.

Using a technique similar to that used in deriving inequality 5, we have: $\delta \hat{y}_2((b_1, 0)) \leq \delta^2 \frac{b_2}{2}$. Now since $\delta^2 \gamma_2((b_1, 0), (0, b_2)) > \delta^2 \frac{b_2}{2}$, we have $\delta^2 \gamma_2((b_1, 0), (0, b_2)) > \delta \hat{y}_2((b_1, 0))$. Thus, player 2 must choose the second option, i.,e, player will reject $(b_1, 0)$ and offers $(0, b_2)$ at Stage 2.

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