A Theory of Value Distribution in Social Exchange Networks

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Abstract: This paper considers the value distribution in an arbitrary exogenously given social exchange network, where any two linked nodes can potentially generate one unit of surplus. A value distribution solution is a mapping that associates a value distribution outcome with each social exchange network. We impose a consistency condition on the value distribution solution. We find that there is a unique value distribution solution that satisfies the consistency condition. As regards some simple networks that are frequently analyzed in the literature, the theoretical prediction based on our solution fits the experimental data very well. Finally, a strategic procedure is proposed to support the solution.

Keywords: Social exchange network; bargaining.

JEL classification: C78

1 Introduction

This paper studies the social exchange network (or simply, network, or graph), where each linked pair can jointly generate a unit of surplus. A node in the network can reach an agreement with at most one neighbor regarding how to divide the joint surplus on their link. That is, the social exchange network considered in this paper is a *negatively* connected network, i.e., the exchange in one relation is contingent on nonexchange in the other (Cook and Emerson (1978)). Examples of such social exchange networks include (i) a network of men and women where each man can marry only one woman, (ii) a network of buyers and sellers where each seller has only one unit of indivisible good and thus can sell his good to at most one buyer, and (iii) a friendship network where each node has one unit amount of time to spend with one of his friends.

A central question in the literature about social exchange networks is how the network structure determines the power or the value of a specific position in the network. In this paper, we propose a method to characterize the value distribution outcome for any given network. The method we propose requires a consistency condition to be imposed on the *value distribution solution*, which is a mapping that associates a value distribution outcome with each network. The consistency condition requires that for any given network, (i) the value that a node obtains is the average of payoffs that the node obtains under all maximum matchings of the network, and (ii) for any given maximum matching and for any two nodes that are matched under the matching, each node's payoff is the node's Nash solution payoff in the bargaining problem where the two nodes' claims are endogenously determined by the values that the two nodes would obtain in the subnetwork where the link between the two nodes is removed.[1] We find that there is one and only one value distribution solution

[1] Such a bargaining problem with claims was first introduced by Chun and Thomson (1992). We allow both feasible claims and infeasible claims (just like Herrero and Villar (2010)). That is, the problem becomes a surplus-sharing problem when the two nodes' claims are feasible (i.e., the sum of the two nodes' claims is less than one), and becomes a rationing problem when the two nodes' claims are infeasible (i.e., the sum of the two nodes' claims exceeds one). A player's claim is the player's subjective expectation about what he should (at least) obtain when he comes to the bargaining table (see, e.g., Chun and Thomson (1992), Gächter and Riedl (2005)).

that satisfies the condition. This unique solution is called the *consistent value distribution solution*, and can be obtained by induction. As regards some simple networks that are frequently analyzed in the literature, the theoretical prediction based on our solution fits the experimental data very well. Moreover, our theory provides a better prediction than the existing theories.

We also propose a strategic procedure that supports the consistent value distribution solution. That is, for any given network, the equilibrium outcome of the strategic procedure coincides with the consistent value solution outcome of the network.

Our paper is mostly related to Cook and Emerson (1978), Cook and Yamagishi (1992) and Kleinberg and Tardos (2008). Cook and Emerson (1978) and Cook and Yamagishi (1992) proposed the *equi-dependence principle*, which requires that the equilibrium bargaining outcome of a matched pair in a network be such that the two parties in the pair depend on each other to the same degree. The outcome that satisfies the equi-dependence principle is called the *balanced outcome* (Kleinberg and Tardos (2008)). In particular, a balanced outcome consists of a matching and a bargaining outcome on the network where any two matched nodes under the matching agree on a division that lies halfway between the two nodes' outside options. A node v's outside option is defined as the maximum of $1-\gamma_u$ among all $u \in U_v$ where U_v is the set of v's neighbors and γ_u is the balanced outcome of node u in the network.

Our consistency condition shares the same spirit with the balanced outcome in the sense that both of them require that a node's payoff under a matching be the split-the-difference outcome between the node's bargaining position and his matched node's bargaining position. The difference is that, in the balanced outcome, a node's bargaining position is represented as the node's *outside option*, while in our paper, a node's bargaining position is represented as the node's *claim*. In the network environment, it seems to make more sense to use a node's claim rather than the node's outside option to describe the node's bargaining position. This is because the meaning of a node's outside option becomes quite vague in the network environment. More precisely, a player's outside option should be the player's guaranteed payoff in case of disagreement. However, in the network environment, when a node breaks the negotiation with his matched node, what the node can obtain depends on how other nodes bargain and what is the new possible match for the node. In the balanced outcome, a node's outside option is defined as the best alternative that the node could obtain from his neighbors under the balanced outcome of the (same) network. This definition seems to be problematic because if a node indeed breaks the negotiation with his matched node, his "best alternative" may also change since he now essentially faces a new network, in which his bargaining power should have decreased as a result of the loss of a bargaining opportunity. On the other hand, in our paper, a node's claim is just a node's subjective expectation about what he should (at least) obtain when he comes to the bargaining table. More precisely, in our definition, a node's claim is what he expects to get under the same value distribution solution when his current link is indeed broken.

Finally, an advantage of our approach is that there exists one and only one value distribution solution that satisfies the consistency condition, while for the balanced outcome, the uniqueness and existence cannot be guaranteed for all networks.

An application of our consistent value distribution solution is that it can be used to rank the power or the values of the nodes in a social exchange network.[2] The node who has a higher consistent value can be regarded as a more powerful or more valuable node than the node who has a lower consistent value. In a friendship network, this means that we can use the consistent value distribution solution to find out who is the most powerful person in the network, and more generally, we can rank the power of the persons in the network. In a network of buyers and sellers, we can use the consistent value distribution solution to find out which seller (or, buyer) has the most valuable position.

This paper is organized as follows. Section 2 presents the basic elements of the model.

[2]Other ranking methods of power of nodes in a network were also proposed in the literature (e.g., Bonacich (1972), Du *et al.* (2013)). The difference is that our theory is more suitable for the ranking of nodes in a *negatively* connected network, while the literature mentioned above is more suitable for the ranking of nodes in a *positively* connected network.

Section 3 discusses the main result. Section 4 studies the strategic procedure that supports the consistent value distribution solution. Concluding remarks are offered in Section 5.

2 Definitions and notation

Let $G = (V, L)$ be a *graph* (or, *network*) with the set of nodes V and the set of links L. Graph G is *undirected* if $(v_i, v_j) \in L \Leftrightarrow (v_j, v_i) \in L$ for any $v_i, v_j \in V$. Graph G is *irreflexive* if $(v_i, v_i) \notin L$ for any $v_i \in V$. Let $\mathcal{G}$ be the set of all undirected and irreflexive graphs.

For any given graph $G \in \mathcal{G}$, we assume that each link in G can potentially generate one unit of surplus. Each node can reach an agreement with at most one neighbor regarding how to divide the one unit of surplus on their incident link. If a node cannot reach an agreement with any neighbor, then the node gets 0.

For a given graph $G = (V, L) \in \mathcal{G}$, let $o(G) = \#|V|$ be the number of nodes in graph G. A *value distribution outcome* on G is a mapping $h : V \to [0, 1]^{o(G)}$, which assigns each node a value in $[0, 1]$. Let $\mathcal{O}(G)$ be the set of all value distribution outcomes for graph G. A *value distribution solution* is a mapping $g : \mathcal{G} \to \cup_{G \in \mathcal{G}} \mathcal{O}(G)$ with $g(G) \in \mathcal{O}(G)$ for any $G \in \mathcal{G}$. That is, a value distribution solution associates a value distribution outcome in $\mathcal{O}(G)$ with each network G in $\mathcal{G}$.

For a given graph $G = (V, L) \in \mathcal{G}$, a *matching* is a one-to-one mapping $m : V \to V$ such that for any $v \in V$, we have (i) $m(m(v)) = v$, and (ii) $m(v) \neq v$ implies that $(v, m(v)) \in L$. Let $S(m, G)$ be the set of all nodes in V that are self matched under the matching m. Let $S^c(m, G) = V \backslash S(m, G)$. For any given $v \in S^c(m, G)$, let $G_{(v,m(v))}$ be the subgraph of G after removing the link between v and $m(v)$, i.e., $G_{(v,m(v))} = (V, L')$ where $L' = L \backslash (v, m(v))$.

Let $NS(a, b)$ be a player's *Nash solution* payoff in the *bargaining problem (with claims)* where the player's claim is a, and the player's opponent's claim is b, where $0 \leq a \leq 1$ and $0 \leq b \leq 1$.[3] That is, $NS(a, b) = a + \dfrac{1 - a - b}{2}$. A player's *claim* is his subjective expectation

[3]The *Nash solution* is formally defined in Herrero and Villar (2010). The Nash solution coincides with the *Nash bargaining solution* (Nash (1950)) when the bargaining problem is a surplus-sharing problem, and

about what he should (at least) obtain when he comes to the bargaining table. Note that the sum of a and b may be less than one, or greater than one. If $a + b \leq 1$, the bargaining problem corresponds to a *surplus-sharing problem*, and if $a + b > 1$, the bargaining problem corresponds to a *rationing problem*.

Define $f(v|m, G, g) = NS(g(G_{(v,m(v))})(v), g(G_{(v,m(v))})(m(v)))$ for $v \in S^c(m, G)$. That is, for any $v \in V$ that is matched with a distinct node $m(v) \in V$, $f(v|m, G, g)$ denotes node v's Nash solution payoff in the bargaining problem where node v's claim is $g(G_{(v,m(v))})(v)$ and node $m(v)$'s claim is $g(G_{(v,m(v))})(m(v))$. In addition, let $f(v|m, G, g) = 0$ for any $v \in S(m, G)$. For a given graph $G \in \mathcal{G}$, we say that a matching m is *maximum* if the set $S^c(m, G)$ is maximal in cardinality among all matchings of G. Let $\mathcal{M}(G)$ be the set of all maximum matchings for graph G. We impose the following condition on the value distribution solution g.

- **Consistency** (**C**): For any graph $G = (V, L) \in \mathcal{G}$ and any $v \in V$, $g(G)(v) = \sum_{m \in \mathcal{M}(G)} \frac{1}{\#|\mathcal{M}(G)|} f(v|m, G, g)$.

Condition **C** requires that (i) the value that a node obtains is the average of payoffs that the node obtains under all maximum matchings of the network[4], and (ii) for each given maximum matching where node v is matched to another node $m(v)$ (i.e., node v is not self-matched), the payoff of node v equals the node's Nash solution payoff in the bargaining problem where node v's claim and node $m(v)$'s claim are endogenously determined by the value distribution solution g in the subgraph of G where the link between v and $m(v)$ is removed.

coincides with the *Nash rationing solution* (Mariotti and Villar (2005)) when the bargaining problem is a rationing problem. Note, however, that our model assumes transferable utility, and thus the Nash solution is simply the equal-gains solution for the surplus-sharing problem and is the equal-losses solution for the rationing problem.

[4]The implicit assumption here is that all maximum matchings occur with equal probabilities.

3 Main result

3.1 Existence and uniqueness

We have the following result regarding the existence and uniqueness of the value distribution solution that satisfies **C**.

Theorem 1. *There exists a unique value distribution solution that satisfies **C**.*

Proof: We prove the theorem by construction. The constructed value distribution solution is denoted by g^*. We construct g^* through the following inductive process.

(i) For any $G = (V, L) \in \mathcal{G}$ with $o(G) = 1$, let $g^*(G)(v) = 0$ for the only node $v \in V$.

(ii) For any $G = (V, L) \in \mathcal{G}$ with $o(G) = 2$ and $\#|L| = 0$, let $g^*(G)(v) = 0$ for any $v \in V$.

(iii) Suppose g^* is determined for any $G \in \mathcal{G}$ with $o(G) \leq k - 1$ and also for any $G \in \mathcal{G}$ with $o(G) = k$ and $\#|L| \leq j$, where k is an integer in $\{2, 3, 4, ...\}$ and j is an integer in $\{0, 1, ..., \frac{k(k-1)}{2}\}$.[5] Now, consider a graph $G = (V, L) \in \mathcal{G}$ with $o(G) = k$ and $\#|L| = j + 1$ if $j < \frac{k(k-1)}{2}$, or a graph $G = (V, L) \in \mathcal{G}$ with $o(G) = k + 1$ and $\#|L| = 0$ if $j = \frac{k(k-1)}{2}$. Define $g^*(G)(v) = \sum_{m \in \mathcal{M}(G)} \frac{1}{\#|\mathcal{M}(G)|} f(v|m, G, g)$ for any $v \in V$. Notice that $g^*(G)(v)$ is well defined because (i) if $o(G) = k$ and $\#|L| = j + 1$, then the calculation of $f(v|m, G, g)$ only involves g^* for the graphs where $o(G) = k$ with $\#|L| = j$, and (ii) if $o(G) = k + 1$ and $\#|L| = 0$, then $f(v|m, G, g) = 0$ for any $v \in V$ and any $m \in \mathcal{M}(G)$.[6]

Given the above inductive process, it is clear that $g^*(G)$ can be defined for any $G \in \mathcal{G}$, and is thus a value distribution solution. It is obvious that g^* satisfies condition **C**. In addition, the above construction makes it clear that g^* is the unique value distribution solution that satisfies condition **C**. □

Theorem 1 shows that there is a unique value distribution solution that satisfies condition **C**. We call this unique solution the *consistent value distribution solution* (or, for simplicity,

[5]Note that for a given graph $G \in \mathcal{G}$ with $o(G) = k$, G contains at most $\frac{k(k-1)}{2}$ links.

[6]There is only one matching for graph G. In this unique matching, all nodes are self matched. This only matching is also a maximum matching.

the *consistent value*).

3.2 Consistent values on some simple graphs

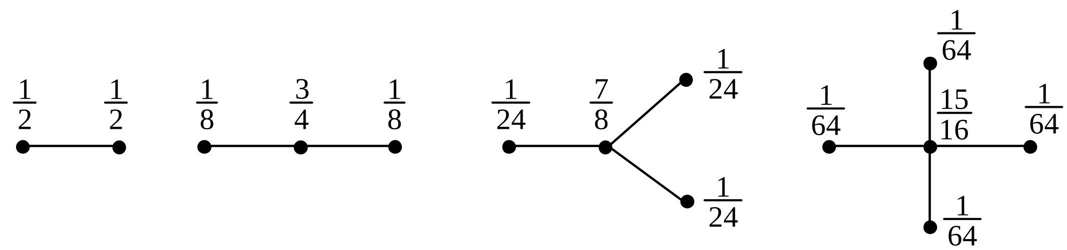

Figure 1: Stars

This subsection illustrates the consistent values on some simple graphs. One group of simple graphs are "stars." Let's consider the star network with one center node and k spoke nodes with $k \geq 1$. If $k = 1$, then there are two nodes in the graph. Obviously, each node obtains a consistent value of 1/2, because there is only one maximum matching and each node's claim is zero under this matching. If $k = 2$, then there are two maximum matchings. For each matching, the center node is matched with a spoke node. In each matching, the center node's claim is 1/2, while the matched spoke node's claim is 0. Thus, the Nash bargaining solution outcome is that the center node obtains 3/4 and the matched spoke node obtains 1/4. The consistent value is thus 1/8 for each of the two spoke nodes and 3/4 for the center node. In general, for the star network with $k \geq 1$ spoke nodes, one can show that the center node obtains a consistent value of $\frac{2^k - 1}{2^k}$ and any spoke node obtains a consistent value of $\frac{1}{k2^k}$. An implication of this result is that the center node's consistent value increases and approaches 1 as the number of spokes increases. This reflects the obvious fact that the center node's "power" increases as the number of the node's neighbors increases.

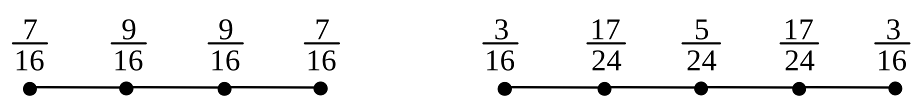

Figure 2: Lines

Next, let's consider lines. Figure 2 illustrates the consistent values on lines with four nodes and five nodes. The two line networks illustrated in Figure 2 are good examples for illustrating the meaning of "well-connected-ness." In the line network with four nodes, each of the two nodes in the middle obtains a value greater than 1/2. The two nodes in the middle are "well-connected" because each node has two neighbors and one of the two neighbors is a "weak" neighbor. However, the middle node in the line network with five nodes also has two neighbors, but the middle node only obtains a value of 5/24. This occurs because the two neighbors of the middle node are "strong" neighbors.

Another group of simple graphs are circles. For the circle with k nodes, it can be verified that the consistent value of each node is 1/2 when k is even, and is $(k-1)/(2k)$ when k is odd. Notice that balanced outcomes for circle networks do not exist when k is odd.

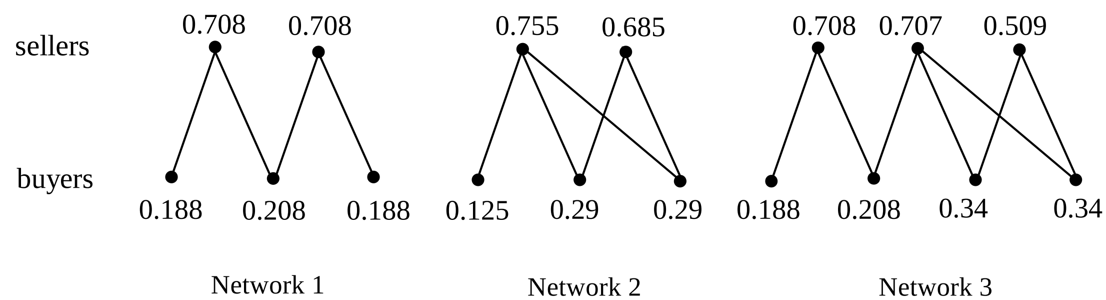

Figure 3: Some bipartite networks

Finally, we consider some bipartite networks. Figure 3 illustrates three bipartite networks, in which sellers are connected with buyers. The first network is essentially the same as the line network with five nodes. In this network, since the sellers are in the short side and the buyers are in the long side, the sellers have more power than the buyers and thus it is not surprising that the sellers obtain higher consistent values than the buyers. The second network illustrates the situation where the two sellers share two commonly linked buyers, but the seller on the left has an additional linked buyer compared with the seller on the right. It is not surprising that the seller on the left obtains a higher consistent value than the seller on the right. In the third network, the seller on the left has two linked buyers, one weak and one strong. The seller on the right, instead, has three linked buyers, all of which are strong. It is not immediately clear which seller's position is more valuable. The consistent value may capture this subtle difference of power between sellers and it shows that the seller position on the left is slightly more valuable than the seller position on the right.[7]

3.3 Comparison with experimental results

This subsection compares the consistent value with experimental results on several simple graphs that are frequently analyzed in the literature. In particular, we focus on the following four networks that were analyzed in Skvoretz and Willer (1993): Branch31, Line4, Stem and Branch32 (refer to Figure 4).[8]

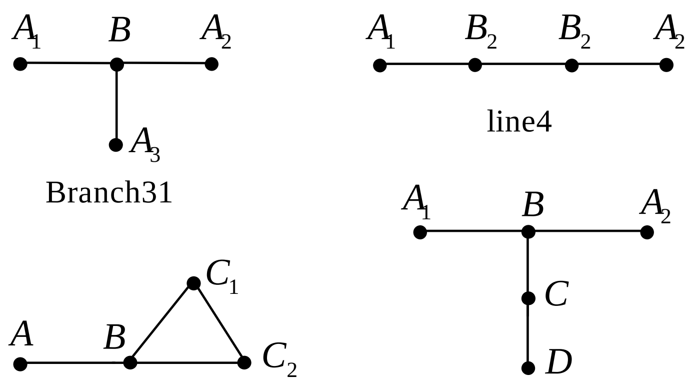

Figure 4: Four frequently-analyzed networks.

The consistent values for node B in each network and node C in Branch32 with the corresponding experimental results are shown in Table 1. The experimental results are obtained from Skvoretz and Willer (1993). Notice that in the table, the total surplus of each

[7]For the three bipartite networks we just analyzed, we can also compare our consistent values with the equilibrium outcomes obtained by Corominas-Bosch (2004), who studied a strategic model in which the buyers and sellers make alternating public offers that may be accepted by any of the linked responders. Corominas-Bosch (2004) showed that if the game starts with the buyers as the proposers, then the unique equilibrium outcome of each of the three networks in Figure 3 is that all buyers obtain 0, and all sellers obtain 1. If, instead, the game starts with the sellers as the proposers, then multiple equilibria may arise. We may also compare our results with that in Manea (2011), who studied the limit equilibrium payoffs of players in a stationary network *where players who have reached agreement will be replaced by new players at the same positions*. For example, for the second network, the limit equilibrium payoff in Manea (2011) is 3/5 for each of the sellers and 2/5 for each of the buyers.

[8]Skvoretz and Willer (1993) analyzed eight common networks. However, the other four networks that we do not analyze in this paper (except the network Kite, for which some existing theory predicts no exchange for some key node) either allow different numbers of exchanges per round for different nodes in the network, or allow different numbers of exchanges per round for different links in the network.

link is rescaled to 24.

Network (position)	Consistent value	Value calculated based on experiments
Branch31 (B)	21	21.6
Line4 (B)	13.5	14.0
Stem (B)	16	15.3
Branch32 (B)	18.7	20.7*
Branch32 (C)	12.5	12.9*

*These values are obtained based on the data on network NT2 in Skvoretz and Willer (1993).

Table 1: A comparison of consistent values and experimental data on four frequently-analyzed networks.

As is clear from the table, the consistent values are close to the experimental data. The only exception is node B in Branch32, for which the consistent value is 18.7, while the experimental value is 20.7. The average difference between the consistent values and the experimental values is 0.84. As regards the four theories considered in Skvoretz and Willer (1993) (i.e., the theories of Core, Equi-dependence, Exchange-Resistance, and Expected Value),[9] the average differences between the predicted values and the experimental values are 2.66, 2.24, 1.98 and 3.96, respectively. This implies that at least for some common networks that are frequently analyzed in the literature, our theory of consistent value seems to provide a better prediction than the existing theories.

4 Supporting the consistent values

The previous section imposed the consistency condition on the value distribution solution and then obtained a unique solution that satisfies the consistency condition. An interesting question is, can we find a meaningful strategic procedure that supports the solution we find? In particular, can we find a bargaining procedure such that for any graph $G \in \mathcal{G}$, the corresponding equilibrium outcome of the procedure coincides with the consistent value on

[9]The prediction of the theory of equi-dependence is exactly the balanced outcome.

the graph?[10]

In this section, we construct a bargaining procedure, which supports the consistent value. In the constructed procedure, at any stage, if the two nodes on a link disagrees on the division of the surplus, then the link is permanently removed from the graph. In addition, the proposer in a disagreed pair will be punished at the next stage in the sense that he can only be a responder at the next stage. If the punished node disagrees with the offer proposed by his matched node at the next stage, then the punished node will be further punished in the sense that he can still only be a responder at the stage after next stage. The above feature of the game ensures that at any stage, for any matched pair, the proposer must make an offer that will be accepted by the responder (as will be clear in the proof of Theorem 2, this implies that the proposer will propose the responder's claim as the offer). Finally, in our procedure, all matched pairs move sequentially. We thus call our procedure the *sequential removal of disagreed link with punishment procedure* (or, for simplicty, *SRDLP procedure*). Let $G \in \mathcal{G}$ be a given graph. The SRDLP procedure is as follows.

- Stage 1:
  - Choose at random one of the maximum matchings in G.
  - For each (non-self) matched pair under the chosen maximum matching, a node is selected at random as the proposer, and the other node in the matched pair is the responder.
  - The matched pairs move sequentially.[11] For each pair, the proposer makes an offer to the corresponding matched responder. The responder then chooses to accept or reject the offer. If all pairs reach agreement (i.e., all responders choose to accept), then the game ends with each node in each matched pair obtaining

[10] A related, but much more difficult question, is whether we can design a *mechanism*, which can be used by the designer to implement the consistent value even when the information about G is unknown to the designer. In the SRDLP procedure defined below, the designer needs to know the set of maximum matchings under the network G in order to enforce the procedure.

[11] The order that the the matched pairs move does not matter, because in equilibrium, all matched pairs reach agreement immediately.

the corresponding payoff proposed by the proposer in the pair and all other nodes (i.e., all self-matched nodes) obtain a payoff of zero. If at any time, some pair disagrees (i.e., the responder of the pair chooses to reject the offer of the proposer), then the game moves to the next stage immediately.

- Stage 2:
  - If the link of the pair that disagrees is the only link in G, then the game ends with all nodes in G obtaining a payoff of zero.
  - If there are more than one link in G, then delete the link between the pair that disagrees and denote the remaining graph as G', which is a subgraph of the original graph. Denote the pair that disagrees at the last stage by (v_i, v_j), where v_i is the proposer. Denote the pair that disagrees at the stage before last stage (if it exists) by (v_l, v_k), where v_l is the proposer. If $v_l = v_j$, let $v^* = v_j$. If $v_l \neq v_j$ or v_l does not exist, then let $v^* = v_i$. Repeat the procedure in Stage 1 with the following two modifications: (i) graph G is replaced by subgraph G'; and (ii) node v^* can only be a responder.[12]

We obtain the following result.

Theorem 2. *For any given graph $G = (V, L) \in \mathcal{G}$, the expected payoff of any node $v \in V$ under the unique SPE of the SRDLP procedure coincides with the consistent value of v.*

Proof: see the appendix.

5 Concluding remarks

This paper considers the value distribution in a given social exchange network. We impose a consistency condition on the the value distribution solution and find that there

[12]This means that for any randomly chosen maximum matching in G', if node v^* is matched with another node in the matching, then node v^* must be the responder, with the other node being the proposer.

is a unique solution, the consistent value, that satisfies the consistency condition. A key element of the consistency condition is that when a node bargains with another node, the node's claim equals the value that the node could obtain from the subnetwork where the link between the two nodes are removed. For some frequently analyzed networks in the literature, the consistent values fit the experimental data very well. Moreover, our theory of the consistent value can be used as a ranking device of the power of nodes in any (negatively connected) social exchange network. We also provide a bargaining procedure that supports the consistent value.

Appendix

Proof of Theorem 2:

The proof is by induction. Let g^* be the consistent value distribution solution.

(i) Suppose $G = (V, L) \in \mathcal{G}$ is such that $\#|L| = 1$.

We have the following two cases.

(a) There is no labeled node in G (here and in the remainder of the proof, the *labeled node* refers to the node that is labeled as a responder).

Since G has only one link, then according to the SRDLP procedure, a node from the only linked pair is chosen at random as the proposer, and the proposer makes an offer to the linked node, which can choose to accept or reject the proposer's offer. Denote the two nodes in the only linked pair by node 1 and node 2. Let $(x, 1-x)$ be the equilibrium offer made by node 1 in the subgame where node 1 is chosen as the proposer, and let $(y, 1-y)$ be the equilibrium offer made by node 2 in the subgame where node 2 is chosen as the proposer.[13] Then we must have: $x = 1$ and $y = 0$. This is because if a responder rejects, then the game will end with all nodes obtaining zero. So, the expected value of node 1 is 1/2, and the expected value of node 2 is also 1/2. Any node in G other than node 1 and node 2 is self-matched and thus obtains a payoff of zero.

[13] Notice that x and y refer to the share of node 1.

So, we have shown that in the case where G is such that $\#|L| = 1$ and there is no labeled node in G, the expected payoff that any node $v \in V$ obtains under the unique SPE of the SRDLP procedure coincides with $g^*(G)(v)$.

(b) There is a labeled node in G.

Let v^* be the labeled node. Let $(v, m(v))$ be the only linked pair. If $v \neq v^*$ and $m(v) \neq v^*$, then a node from v and $m(v)$ is chosen as the proposer at random, and the expected payoff of each node is $1/2$ (the reasoning here is similar to that in (a) and is omitted).

If $v = v^*$ or $m(v) = v^*$, then for the pair $(v^*, m(v^*))$, node $m(v^*)$ must be the proposer because v^* can only be a responder. In addition, $m^*(v)$ must propose 0 to v^*, which v^* accepts in equilibrium. So, the payoff of v^* is zero.

In addition, according the SRDLP procedure, for any $v \in V$ that is self-matched, the payoff of v must be zero.

So, we have shown that in the case where G is such that $\#|L| = 1$ and there is a labeled node in G, the expected payoff of the labeled node must be zero, and the expected payoff of any node $v \in V$ that is not linked to the labeled node is $g^*(G)(v)$.

(ii) Suppose we have proved that (a) for any $G = (V, L) \in \mathcal{G}$ with $\#|L| \leq k$ where $k \in \{1, 2, ..., \dfrac{o(G)(o(G)-1)}{2} - 1\}$ and there is no labeled node in G, the expected payoff of any node $v \in V$ under the unique SPE of the SRDLP procedure coincides with the consistent value $g^*(G)(v)$; and (b) for any $G = (V, L) \in \mathcal{G}$ with $\#|L| \leq k$ where $k \in \{1, 2, ..., \dfrac{o(G)(o(G)-1)}{2} - 1\}$ and there is a labeled node in G, the expected payoff of the labeled node under the unique SPE of the SRDLP procedure is zero, and the expected payoff of any node $v \in V$ that is not linked to the labeled node under the unique SPE of the SRDLP procedure coincides with the consistent value $g^*(G)(v)$. We next show that the above two facts hold for any $G = (V, L) \in \mathcal{G}$ with $\#|L| = k + 1$.

(a) There is no labeled node in G.

Let m be the chosen matching. Suppose it is the turn for $(v, m(v))$ to move. If v is the proposer, then his offer must be $(1 - g^*(G_{(v,m(v))})(m(v)), g^*(G_{(v,m(v))})(m(v)))$,[14] which node $m(v)$ chooses to accept in equilibrium. This is because if $m(v)$ rejects, then the game moves to the next stage with v being labeled as responder, and thus $m(v)$ obtains a payoff of $g^*(G_{(v,m(v))})(m(v))$ at the next stage by the inductive assumption.[15] Similarly, if $m(v)$ is the proposer, then his offer must be $(g^*(G_{(v,m(v))})(v), 1 - g^*(G_{(v,m(v))})(v))$, which node v chooses to accept in equilibrium. So, the expected payoff of node v is $\frac{1}{2}(1 - g^*(G_{(v,m(v))})(m(v)) + g^*(G_{(v,m(v))})(v))$, which coincides with $NS(g^*(G_{(v,m(v))})(v), g^*(G_{(v,m(v))})(m(v)))$. Similarly, the expected payoff of $m(v)$ coincides with $NS(g^*(G_{(v,m(v))})(m(v)), g^*(G_{(v,m(v))})(v))$.

If a node is not in any matched pair under the matching m (i.e., $v \in S(m, G)$), then the node obtains a payoff of zero.

Given the analysis above and the fact that all maximum matchings are chosen with equal probabilities, it is obvious that the expected payoff of any node $v \in V$ is $\sum_{m \in \mathcal{M}(G)} \frac{1}{\#|\mathcal{M}(G)|} f(v|m, G, g^*)$, where $f(v|m, G, g^*) = NS(g(G_{(v,m(v))})(v), g^*(G_{(v,m(v))})(m(v)))$ for $v \in S^c(m, G)$ and $f(v|m, G, g^*) = 0$ for $v \in S(m, G)$. That is, the expected payoff of any node $v \in V$ coincides with $g^*(G)(v)$.

(b) There is a labeled node in G.

Let v^* be the labeled node. Let m be the chosen matching. Suppose it is the turn for $(v, m(v))$ to move, where $v \neq v^*$ and $m(v) \neq v^*$. If v is the proposer, then his offer must be $(1 - g^*(G_{(v,m(v))})(m(v)), g^*(G_{(v,m(v))})(m(v)))$, which node $m(v)$ chooses to accept in equilibrium.[16] Similarly, if $m(v)$ is the proposer, then his offer must be $(g^*(G_{(v,m(v))})(v), 1 - g^*(G_{(v,m(v))})(v))$, which node v chooses to accept in equilibrium. So, the expected payoff of node v is $\frac{1}{2}(1 - g^*(G_{(v,m(v))})(m(v)) + g^*(G_{(v,m(v))})(v))$, which coincides with $NS(g^*(G_{(v,m(v))})(v), g^*(G_{(v,m(v))})(m(v)))$. Similarly, the expected payoff of $m(v)$ coincides with $NS(g^*(G_{(v,m(v))})(m(v)), g^*(G_{(v,m(v))})(v))$.

[14] Recall that $G_{(v,m(v))}$ is the subgraph of G with the link between $(v, m(v))$ being removed.

[15] Notice that the subgraph $G_{(v,m(v))}$ contains k links, and $m(v)$ is not linked to v in $G_{(v,m(v))}$.

[16] The reasoning here is similar to that in (a). That is, if $m(v)$ rejects, then the game moves to the next stage with the link between $(v, m(v))$ removed and v being labeled as a responder, and thus $m(v)$ obtains $g^*(G_{(v,m(v))})(m(v))$ at the next stage by the inductive assumption.

Suppose it is the turn for $(v^*, m(v^*))$ to move. Then $m(v^*)$ must be the proposer, and $m(v^*)$ must propose 0 to v^*, which v^* accepts in equilibrium. This is because if v^* rejects, then the game moves to the next stage with the link between $(v^*, m(v^*))$ removed and v^* again being labeled as a responder, and thus v^* obtains zero at the next stage according to the inductive assumption.

If a node is self-matched under the matching m (i.e., $v \in S(m, G)$), then the node obtains a payoff of zero.

Given the analysis above and the fact that all maximum matchings are chosen with equal probabilities, it is obvious that the expected payoff of a labeled node is zero, and the expected payoff of any node $v \in V$ that is not linked to the labeled node is $g^*(G)(v) = \sum_{m \in \mathcal{M}(G)} \frac{1}{\#|\mathcal{M}(G)|} f(v|m, G, g^*)$.

In equilibrium, there will be no labeled node, because a labeled node will obtain a payoff of zero. The above inductive process thus implies that for any given graph $G = (V, L) \in \mathcal{G}$, the expected payoff of any node $v \in V$ under the unique SPE of the SRDLP procedure coincides with the consistent value of v. □

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