Unbounded Returns and the Possibility of Credit Rationing: A Note on the Stiglitz-Weiss and Arnold-Riley Models

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Arnold and Riley (2009) find that in the credit rationing model of Stiglitz and Weiss (1981), the expected revenue of a lender as a function of the loan rate cannot be globally hump-shaped, and thus credit rationing is hard to explain using the Stiglitz-Weiss model. However, Arnold and Riley base their analysis on the assumption that there is an upper bound of the returns of borrowers' projects. We find that if unbounded returns of borrowers' projects are allowed, then a lender's expected revenue in the Stiglitz-Weiss model can in fact be globally hump-shaped. This also implies that credit rationing (with one equilibrium loan rate) can only arise in markets where the returns from investment are highly volatile.

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1 Introduction

Credit rationing refers to the situation where the equilibrium demand for loanable funds in financial markets exceeds the equilibrium supply. In a seminal paper, Stiglitz and Weiss (1981) (hereafter SW) explain credit rationing using adverse selection. SW have had a far-reaching influence on the literature on adverse selection, which has improved our understanding of credit rationing and many related issues (see Arnold and Riley (2009) for a brief survey).

The main idea of SW is as follows. As the lender increases his loan rate, there exist two opposite effects on the lender's expected revenue. The first is that the lender will earn more if borrowers' projects turn out to be successful. The second is adverse selection, in which low-risk borrowers will be crowded out. If the second effect dominates the first effect when the loan rate is high—such that the lender's expected revenue as a function of the loan rate is globally hump-shaped—then the lender may prefer to set a relatively low loan rate, which causes credit rationing.

Arnold and Riley (2009) (hereafter AR) point out that in SW a lender's expected revenue function *cannot* be globally hump-shaped if project returns are bounded. A key observation of AR is that when the loan rate is high enough such that only the most risky borrower is willing to borrow, that borrower must be a marginal borrower and thus obtains an expected payoff of zero. In this case, the lender will seize all the expected surplus of the borrower's project, which implies that the lender's expected revenue will reach its maximum. This implies that when the loan rate is sufficiently high, the lender's expected revenue must be an increasing function of the loan rate, which in turn implies that the lender's expected revenue function cannot be globally hump-shaped.

An implication of AR is that "rationing is hard to explain using this [SW] model" (AR, p. 2013). However, it should be noted that AR's analysis is based on the assumption that projects have bounded returns (in particular, AR focus on projects whose outcomes have bounded support). That is, AR rule out the possibility that projects have unbounded positive returns. Unbounded positive returns are implicitly allowed in SW (p. 395) by assuming infinity as the upper bound when calculating expected returns. Hence by reconsidering this previously omitted case, we offer the possibility to identify situations where the SW model may correctly explain credit rationing, and thus lead us to a better understanding of SW.

More precisely, in AR, a lender faces a large market in which there is a continuum of borrowers where each borrower has one unit of demand for funds. Borrowers may differ in the returns and thus the risk of their projects, and the risk (or type) of each borrower is (independently) drawn from a common distribution. It is assumed in AR that projects have bounded returns—which essentially implies that there exists a type of borrower who is more risky than any other type of borrower. Our paper relaxes this assumption and allows unbounded returns. For simplicity, we assume that a borrower's project can only have two possible outcomes: failure or success. The realized return of the borrower's project in case of failure is fixed and the same for all types. The realized return in case of success differs across types and is *unbounded* (i.e., there does not exist a uniform upper bound such that the realized returns of *all* types' projects in case of success are less than this bound). We assume that for all types of a borrower, their projects have the same expected return, so that if the realized return of a type's project when successful is high, then the probability that the project will be successful must be small (noting that a project that has a low success probability but a high return when successful is more risky than a project that has a high success probability but a low return when successful).[1]

Our main finding is that when unbounded returns are allowed, a lender's expected revenue can be globally hump-shaped. The intuition is that if there are unbounded returns, then no matter how high the loan rate, there always exist borrowers whose projects are more risky than the marginal borrower at that loan rate. The marginal borrower will still obtain an expected payoff of zero, but borrowers who are more risky than the marginal borrower will obtain positive payoffs, which implies that the lender's expected revenue will be bounded

[1] A project is more risky than another project if the distribution of the return of the former project is a mean preserving spread of the distribution of the return of the latter project.

away from the common expected return of projects, no matter how high the loan rate is. This implies that the lender's expected revenue may be globally hump-shaped. Our numerical examples demonstrate that if the borrower's type is distributed in a way such that the tail of the distribution is heavy—such as a Pareto distribution or a (truncated) t-distribution—then the lender's expected revenue can be globally hump-shaped.

2 Analysis

We assume that there is a unit mass of loan applicants[2] and a bank (hereafter a lender). Each applicant has a project that needs one unit of fund. The (gross) return on the project of a type t applicant is $\tilde{y}(t) = \mu + \tilde{z}(t)$, where $\mu > 0$. For simplicity, we assume that $\tilde{z}(t)$ is a two-outcome random variable. In particular, $\tilde{z}(t) = -\mu$ with probability $1-p(t)$ and $\tilde{z}(t) = t$ with probability $p(t)$. We assume that an applicant's type is the applicant's private information. However, it is common knowledge that any applicant's type is (independently) drawn from a common distribution with p.d.f. $g(t)$ and c.d.f $G(t)$. In addition, the support of $g(t)$ is $[x, \infty)$ (where $x \geq 0$). We assume that all types of projects have the same expected return, and their common expected return equals to μ. This implies that $-\mu(1 - p(t)) + tp(t) = 0$, i.e., $p(t) = \dfrac{\mu}{\mu + t}$ for any $t \in [x, \infty)$. Each applicant has a collateral $C \geq 0$. For simplicity, we assume that $C \leq \mu$. For a successful loan applicant (hereafter a borrower), his (ex post) payoff is $u(z, R) = \max\{\mu + z - R, -C\}$, where R is the (gross) loan rate charged by the lender and z is the realized value of $\tilde{z}(t)$. The expected payoff for a type t borrower is thus:

[2]This assumption—that there are many loan applicants for each unit of fund—follows from Stiglitz and Weiss (1981) and Arnold and Riley (2009), and is appropriate when the market is fully competitive or sufficiently large.

$$
\begin{aligned}
U(t,R) &= E_{\tilde{z}(t)} u(\tilde{z}(t), R) \\
&= E_{\tilde{z}(t)} \max\{\mu + \tilde{z}(t) - R, -C\} \\
&= p(t) \max\{\mu + t - R, -C\} + (1 - p(t)) \max\{-R, -C\} \\
&= \frac{\mu}{\mu + t} \max\{\mu + t - R, -C\} + \frac{t}{\mu + t} \max\{-R, -C\}.
\end{aligned}
$$

Since $u(z, R)$ is a convex function in z and the expected return on the project of any type of borrower is the same, the borrower with type t must obtain a higher expected payoff than the borrower with type t' if $t > t'$.

We now derive the *expected revenue* of the lender, which is denoted by $V(R)$. We first consider the case where $R < C$. In this case, the loan rate is so low that the borrower will never lose all of his collateral. The borrower's expected payoff is thus:

$$
\begin{aligned}
U(t,R) &= \frac{\mu}{\mu + t} \max\{\mu + t - R, -C\} + \frac{t}{\mu + t} \max\{-R, -C\} \\
&= \frac{\mu}{\mu + t}(\mu + t - R) + \frac{t}{\mu + t}(-R) \\
&= \mu - R.
\end{aligned}
$$

Note that this payoff is independent of the borrower's type and is positive under the assumption that $\mu \geq C$. This implies that all borrower types are willing to borrow from the lender, and there is no screening. Since the sum of the borrower's payoff and the lender's payoff must equal μ, we thus have $V(R) = \mu - U(t, R) = R$.

We now consider the case where $R \geq C$. In this case, the loan rate is sufficiently high so that if the return on the borrower's project is low, the borrower will lose all of his collateral. However, if the return on the borrower's project is high, then the borrower may or may not lose all of his collateral, depending on his type. We use $\theta(R)$ to denote the lowest type whose expected payoff is equal to or greater than zero. Then for any $t \geq \theta(R)$, the type t borrower cannot earn a negative payoff when the realization of the return is high (otherwise, his expected payoff will be negative). So, for any $t \geq \theta(R)$, we have:

$$
\begin{aligned}
U(t,R) &= \frac{\mu}{\mu+t}\max\{\mu+t-R,-C\} + \frac{t}{\mu+t}\max\{-R,-C\} \\
&= \frac{\mu}{\mu+t}(\mu+t-R) + \frac{t}{\mu+t}(-C) \\
&= \mu - \frac{\mu}{\mu+t}R - \frac{t}{\mu+t}C \\
&= \mu - R + \frac{t}{\mu+t}(R-C).
\end{aligned}
$$

Note that $U(t,R) = 0$ implies that $t = \frac{\mu(R-\mu)}{\mu-C}$. Therefore, $\theta(R) = \max\{x, \frac{\mu(R-\mu)}{\mu-C}\}$. For a given loan rate R, only applicants with types no less than $\theta(R)$ are willing to borrow. Therefore, the expected revenue of the lender is:

$$
\begin{aligned}
V(R) &= \mu - E_t(U(t,R)|t \geq \theta(R)) \\
&= \mu - E_t[\mu - R + \frac{t}{\mu+t}(R-C)|t \geq \theta(R)] \\
&= R - (R-C)\int_{\theta(R)}^{\infty} \frac{t}{\mu+t}\frac{g(t)}{1-G(\theta(R))}dt.
\end{aligned}
$$

Note that if $R \leq \mu + \frac{x(\mu-C)}{\mu}$, then $\frac{\mu(R-\mu)}{\mu-C} \leq x$, which implies that $\theta(R) = x$. So, if $R \leq \mu + \frac{x(\mu-C)}{\mu}$, then $V(R) = R - (R-C)m = (1-m)R + mC$, where $m = \int_x^{\infty} \frac{t}{\mu+t}g(t)dt = E(\frac{t}{\mu+t}) < 1$.

In summary, we have:

Proposition 1. *The expected revenue of the lender is*

$$
V(R) = \begin{cases}
R & \text{if } R < C \text{ (no default);} \\
(1-m)R + mC & \text{if } C \leq R \leq \mu + \frac{x(\mu-C)}{\mu} \text{ (some default)} \\
R - (R-C)\int_{\theta(R)}^{\infty} \frac{t}{\mu+t}\frac{g(t)}{1-G(\theta(R))}dt & \text{if } R > \mu + \frac{x(\mu-C)}{\mu} \text{ (adverse selection)}
\end{cases}
$$

According to Proposition 1, it is obvious that $V(R)$ is increasing in R for $0 \leq R \leq \mu + \frac{x(\mu-C)}{\mu}$. However, the shape of $V(R)$ for $R > \mu + \frac{x(\mu-C)}{\mu}$ crucially depends on the distribution of borrower types. We next give two examples of distribution of borrower types and illustrate that $V(R)$ is globally hump-shaped in these two cases.

Examples. Assume that $\mu = 1.5$ and $C = 0.5$. Figure 1 illustrates $V(R)$ for cases where (i) t follows a Pareto distribution with $\alpha = 1$ and $x = 1$ (i.e., $g(t) = \frac{\alpha x^{\alpha}}{t^{\alpha+1}} = \frac{1}{t^2}$ for $t \geq 1$), and (ii) t follows a truncated t distribution with degree of freedom $= 1$ and $x = 0$ (i.e., $g(t) = \frac{2}{\pi(1+t^2)}$ for $t \geq 0$). In both cases, $V(R)$ is globally hump-shaped. With the Pareto distribution, $V(R)$ decreases once adverse selection kicks in. However, in the case of the truncated t distribution, when adverse selection starts to play a role, $V(R)$ first increases a little bit before the subsequent drop .

More formally, it can be shown that in both cases, $V(R)$ will converge to $\frac{\mu + C}{2} = 1$ as R goes to infinity (see the Appendix for the proof). □

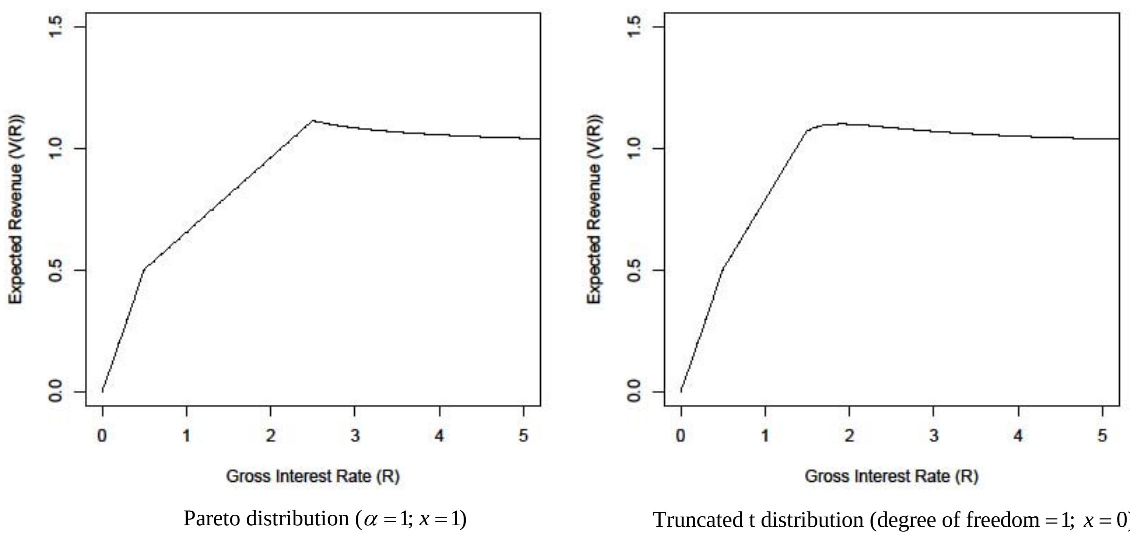

Figure 1

The analysis above only provides some examples to show that when unbounded returns of borrowers' projects are allowed, the expected revenue $V(R)$ of the lender can be globally hump-shaped. It should be noted that we are not intended to provide sufficient and necessary conditions of distributions under which $V(R)$ is globally hump-shaped.[3] Instead, the main

[3]It is a very complicated task to find the sufficient and necessary conditions. However, a general observation is that for heavy-tailed distributions of t, it is likely that $V(R)$ is decreasing for sufficiently large R (and thus $V(R)$ is globally hump-shaped), while for thin-tailed distributions of t, it is likely that $V(R)$ is increasing for $R > \mu + \frac{x(\mu - C)}{\mu}$ (and thus $V(R)$ is globally increasing; for example, it can be verified that if t follows (i) a truncated normal distribution of $N(0,1)$ with $x = 0$ (i.e., $g(t) = \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ for $t \geq 0$), or (ii)

purpose of this paper is simply to establish a possibility in which the SW model can explain credit rationing correctly, and we indeed find such a possibility (i.e., when borrowers' projects have unbounded returns).

3 Conclusion

Arnold and Riley (2009) find that in the credit rationing model of Stiglitz and Weiss (1981), a lender's expected revenue as a function of the loan rate can never be globally hump-shaped, and thus credit rationing is hard to explain using the SW model. A main argument of AR is that "...the SW paper, despite its crucial role in financial economics, is still in need of some clarification" (AR, p. 2012). Our paper answers this question by providing a situation in which the SW model can explain credit rationing correctly. In particular, we find a gap in AR by allowing unbounded returns about borrowers' projects, which is a situation ruled out by AR but implicitly allowed in SW. We find that a lender's expected revenue can be globally hump-shaped when unbounded returns are allowed. This also implies that the SW model can explain credit rationing when banks face a market in which the returns from investment are highly volatile.

References

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an exponential distribution with $\lambda = 1$ and $x = 0$ (i.e., $g(t) = \lambda e^{-\lambda t} = e^{-t}$ for $t \geq 0$), then $V(R)$ is globally increasing). However, the above observation is not always true. It can be verified that some heavy-tailed distributions of t can also yield a globally increasing $V(R)$. For example, if t follows a log-normal distribution $lnN(0,1)$ (i.e., $g(t) = \frac{1}{t\sqrt{2\pi}} e^{-\frac{(lnt)^2}{2}}$ for $t > 0$), then $V(R)$ will be globally increasing.

Appendix

Proof in Examples: Suppose t follows a Pareto distribution with $\alpha = 1$ and $x = 1$. That is, $g(t) = \frac{\alpha x^\alpha}{t^{\alpha+1}} = \frac{1}{t^2}$ for $t \geq 1$, and $G(t) = 1 - (\frac{x}{t})^\alpha = 1 - \frac{1}{t}$. We will show that $\lim_{R\to\infty} V(R) = \frac{\mu + C}{2} = 1$.

Note that when R is sufficiently large, $V(R)$ can be calculated as follows:

$$
\begin{aligned}
V(R) &= R - (R-C)\int_{\theta(R)}^{\infty} \frac{t}{\mu+t}\frac{g(t)}{1-G(\theta(R))}dt \\
&= R - (R-C)\frac{1}{1-G(\theta(R))}\int_{\theta(R)}^{\infty} \frac{t}{\mu+t}g(t)dt \\
&= R - (R-C)\frac{\mu(R-\mu)}{\mu-C}\int_{\theta(R)}^{\infty} \frac{t}{\mu+t}\frac{1}{t^2}dt \qquad (\theta(R) = \frac{\mu(R-\mu)}{\mu-C} \text{ when } R \text{ is large}) \\
&= R - (R-C)\frac{\mu(R-\mu)}{\mu-C}\frac{1}{\mu}\int_{\theta(R)}^{\infty}(\frac{1}{t} - \frac{1}{\mu+t})dt \\
&= R - (R-C)\frac{R-\mu}{\mu-C}(\ln t - \ln(\mu+t))|_{t=\theta(R)}^{\infty} \\
&= R - (R-C)\frac{R-\mu}{\mu-C}(-\ln\frac{\theta(R)}{\mu+\theta(R)}) \\
&= R + (R-C)\frac{R-\mu}{\mu-C}\ln\frac{R-\mu}{R-C}.
\end{aligned}
$$

So, we have:

$$
\begin{aligned}
&\lim_{R\to\infty} V(R) \\
&= \lim_{R\to\infty}[R + (R-C)\frac{R-\mu}{\mu-C}\ln\frac{R-\mu}{R-C}] \\
&= \lim_{R\to\infty}[C + (R-C)[1 + \frac{R-\mu}{\mu-C}\ln\frac{R-\mu}{R-C}]] \\
&= C + \lim_{R\to\infty}\frac{1 + \frac{R-\mu}{\mu-C}\ln\frac{R-\mu}{R-C}}{1/(R-C)} \\
&= C + \lim_{R\to\infty}\frac{\frac{1}{\mu-C}[(R-\mu)(\frac{1}{R-\mu} - \frac{1}{R-C}) + \ln\frac{R-\mu}{R-C}]}{-1/(R-C)^2} \qquad \text{(L'Hospital's rule)} \\
&= C + \lim_{R\to\infty}\frac{\frac{1}{\mu-C}[-\frac{\mu-C}{(R-C)^2} + (\frac{1}{R-\mu} - \frac{1}{R-C})]}{2/(R-C)^3} \qquad \text{(L'Hospital's rule)}
\end{aligned}
$$

$$= C + \lim_{R\to\infty} \frac{\frac{\mu - C}{(R-\mu)(R-C)^2}}{2/(R-C)^3}$$

$$= C + \frac{\mu - C}{2}$$

$$= \frac{\mu + C}{2}$$

$$= 1.$$

The proof for the case where t follows a truncated t-distribution is similar and is omitted.