

Unbounded Risk and the Possibility of Credit Rationing: A Note on the Stiglitz-Weiss and Arnold-Riley Models

Hengheng Lu

School of Management, Fudan University

Kang Rong*

School of Economics, Shanghai University of Finance and Economics (SUFU)
Key Laboratory of Mathematical Economics (SUFU), Ministry of Education, China

April 25, 2017

Arnold and Riley (2009) find that in the credit rationing model of Stiglitz and Weiss (1981), the expected revenue of a lender as a function of the loan rate can never be globally hump-shaped, and thus credit rationing is hard to explain using the Stiglitz-Weiss model. However, Arnold and Riley implicitly assume that there is an upper bound of the risk of borrowers' projects. We find that if unbounded risk (or equivalently, unbounded returns) of borrowers' projects is allowed, then a lender's expected revenue in the Stiglitz-Weiss model can in fact be globally hump-shaped. This also implies that credit rationing (with one equilibrium loan rate) can only arise in markets where the returns from investment are highly volatile. More generally, our analysis casts some new insight on the literature of adverse selection by emphasizing the role of type distribution of the informed party in the analysis of the effect of adverse selection.

Keywords: Credit rationing, unbounded risk, adverse selection.

JEL classification: D82, D45, G21.

*We thank Sambuddha Ghosh, John Riley, and Steve Williams for helpful discussions and comments.

1 Introduction

Credit rationing refers to the situation where the equilibrium demand for loanable funds in financial markets exceeds the equilibrium supply. In a seminal paper, Stiglitz and Weiss (1981) (hereafter SW) explain credit rationing using adverse selection. SW have had a far-reaching influence on the literature on adverse selection, which has improved our understanding of credit rationing and many related issues (see Arnold and Riley (2009) for a brief survey).

The main idea of SW is that a lender uses the loan rate as a screening device to distinguish high-risk and low-risk borrowers. As the lender increases his loan rate, there exist two opposite effects on the lender's expected revenue. The first is that the lender will earn more if borrowers' projects turn out to be successful. The second is adverse selection, in which low-risk borrowers will be crowded out. If the second effect dominates the first effect when the loan rate is high—such that the lender's expected revenue as a function of the loan rate is globally hump-shaped—then the lender may prefer to set a relatively low loan rate, which causes credit rationing.

Arnold and Riley (2009) (hereafter AR) perceptively point out that in SW, a lender's expected revenue function can *never* be globally hump-shaped. A key observation of AR is that when the loan rate is high enough such that only the most risky borrower is willing to borrow, that borrower must be a marginal borrower and thus obtains an expected payoff of zero. In this case, the lender will seize all the expected surplus of the borrower's project, which implies that the lender's expected revenue will reach its maximum. This implies that when the loan rate is sufficiently high, the lender's expected revenue must be an increasing function of the loan rate, which in turn implies that the lender's expected revenue function *cannot* be globally hump-shaped.

An implication of AR is that “rationing is hard to explain using this [SW] model” (AR, p. 2013). However, AR's finding is based on the implicit assumption that borrowers' projects have bounded risk in the sense that there exists a most risky borrower in the market. In

reality, *unbounded risk* may arise when people are not sure what is the upper bound of risk of borrowers' projects.¹ It is thus natural to reconsider the validity of the SW model by allowing unbounded risk. Such consideration may offer the possibility to identify situations where the SW model can correctly explain credit rationing, and thus lead us to a better understanding of SW.

More precisely, in AR, a lender—who has a unit of loanable fund—faces a large market in which there is a continuum of borrowers where each borrower has one unit of demand for funds. Borrowers may differ in the risk of their projects, and the risk (or type) of each borrower is (independently) drawn from a common distribution. It is implicitly assumed in AR that there exists a type of borrower who is more risky than any other type of borrower. Our paper relaxes this assumption and allows unbounded risk. For simplicity, we assume that a borrower's project can only have two possible outcomes: failure or success. The realized return of the borrower's project in case of failure is fixed and the same for all types. The realized return in case of success differs across types and is *unbounded* (i.e., there does not exist a uniform upper bound such that the realized return of *any* type's project in case of success is less than this bound). We assume that for all types of a borrower, their projects have the same expected return, so that if the realized return of a type's project when successful is high, then the probability that the project will be successful must be small. Note that a project that has a low success probability but a high return when successful is more risky than a project that has a high success probability but a low return when successful.² The assumption that a borrower's project has unbounded realized return in case of success—which also implies unbounded risk in our model—is natural. For example, the return on a stock, which usually reflects the realized return of the underlying project, is usually modeled as being unbounded (see also footnote 1).

¹Moreover, unbounded risk is often a result of *unbounded return*, which is a basic assumption in Finance (for example, a common way to model the behavior of the return on a stock is to assume that it follows a geometric Brownian motion, which implies that the return on the stock in a given period follows a normal distribution).

²A project is more risky than another project if the distribution of the return of the former project is a mean preserving spread of the distribution of the return of the latter project.

Our main finding is that when unbounded risk is allowed, a lender’s expected revenue can be globally hump-shaped. The intuition is that if there is unbounded risk, then no matter how high the loan rate, there always exist borrowers whose projects are more risky than the marginal borrower at that loan rate. The marginal borrower will still obtain an expected payoff of zero, but borrowers who are more risky than the marginal borrower will obtain positive payoffs, which implies that the lender’s expected revenue will be bounded away from the common expected return of projects, no matter how high the loan rate is. This implies that the lender’s expected revenue may be globally hump-shaped. Our numerical examples demonstrate that if the borrower’s type is distributed in a way such that the tail of the distribution is heavy—such as a Pareto distribution or a (truncated) t -distribution—then the lender’s expected revenue can be globally hump-shaped.

Our finding suggests that credit rationing (with one equilibrium loan rate) can only arise in markets where the returns from investment are highly volatile.³ The intuition is as follows. In general, the loan rate set by the lender serves as a screening device. For any loan rate set by the lender, the lender can infer that the pool of the borrowers who are willing to borrow consists of borrowers who are more risky than the “marginal borrower” at the loan rate. As the lender increases the loan rate, this pool of borrowers become more risky (because the “marginal borrower” becomes more risky); however, the variance (or variation) of the risk of borrowers in this pool becomes smaller *if there is an upper bound about the risk of borrowers*. This second effect is beneficial to the lender,⁴ and it may alleviate the negative effect of the lender’s facing a more risky pool of borrowers and thus induce the lender to set an efficient loan rate (which clears the market), rather than set a low loan rate (which causes credit rationing). On the other hand, *if the lender faces a market where the returns from investment are unbounded*, increasing the loan rate will hardly decrease the variation

³Credit rationing with two equilibrium loan rates can arise in AR model; however, as pointed out by AR, there is actually no credit rationing at the high rate (although there is credit rationing at the low rate).

⁴More precisely, as the loan rate increases, the pool of borrowers who are willing to borrow become more “homogenous,” and the average type of the borrowers in this pool become closer to the marginal borrower. Note that the lender can always extract full surplus from the marginal borrower, so as the average type becomes closer to the marginal borrower, the lender will be “better off.”

of the risk of the borrowing pool, and thus the lender may prefer to set a relatively low loan rate, which causes credit rationing.

Our analysis above emphasizes the *composition effect* of adverse selection, which is often overlooked in the literature. In particular, as the uninformed party increases the price, the composition of types of remaining informed parties become less “heterogeneous,” which is beneficial to the lender. However, the extent at which the composition effect benefits the lender may crucially depend on the distribution of types of informed parties. The model in AR is a good example in showing that the composition effect is large enough so that it significantly alleviates the adverse selection effect and a Walrasian equilibrium outcome arises.

2 Analysis

We assume that there is a unit mass of loan applicants⁵ and a bank (hereafter a lender). The lender has one unit of loanable fund, and each applicant has a project that needs one unit of fund. The (gross) return on the project of a type t applicant is $\tilde{y}(t) = \mu + \tilde{z}(t)$, where $\mu > 0$. For simplicity, we assume that $\tilde{z}(t)$ is a two-outcome random variable. In particular, $\tilde{z}(t) = -\mu$ with probability $1 - p(t)$ and $\tilde{z}(t) = t$ with probability $p(t)$. We assume that an applicant’s type is the applicant’s private information. However, it is common knowledge that any applicant’s type is (independently) drawn from a common distribution with p.d.f. $g(t)$ and c.d.f. $G(t)$. In addition, the support of $g(t)$ is $[x, \infty)$ (where $x \geq 0$). We assume that all types of projects have the same expected return, and their common expected return equals to μ . This implies that $-\mu(1 - p(t)) + tp(t) = \mu$, i.e., $p(t) = \frac{\mu}{\mu + t}$ for any $t \in [x, \infty)$. Each applicant has a collateral $C \geq 0$. For simplicity, we assume that $C \leq \mu$. For a successful loan applicant (hereafter a borrower), his (ex post) payoff is $u(z, R) = \max\{\mu + z - R, -C\}$, where R is the (gross) loan rate charged by the lender and z is the realized return of the

⁵This assumption—that there are many loan applicants for each unit of fund—follows from Stiglitz and Weiss (1981) and Arnold and Riley (2009), and is appropriate when that the market is fully competitive or sufficiently large.

borrower's project. The expected payoff for a type t borrower is thus:

$$\begin{aligned}
U(t, R) &= E_{\tilde{z}(t)} u(\tilde{z}(t), R) \\
&= E_{\tilde{z}(t)} \max\{\mu + \tilde{z} - R, -C\} \\
&= p(t) \max\{\mu + t - R, -C\} + (1 - p(t)) \max\{-R, -C\} \\
&= \frac{\mu}{\mu + t} \max\{\mu + t - R, -C\} + \frac{t}{\mu + t} \max\{-R, -C\}.
\end{aligned}$$

Since $u(z, R)$ is a convex function in z and the expected return on the project of any type of borrower is the same, the borrower with type t must obtain a higher expected payoff than the borrower with type t' if $t > t'$.

We now derive the *expected revenue* of the lender, which is denoted by $V(R)$. We first consider the case where $R < C$. In this case, the loan rate is so low that the borrower will never lose all of his collateral. The borrower's expected payoff is thus:

$$\begin{aligned}
U(t, R) &= \frac{\mu}{\mu + t} \max\{\mu + t - R, -C\} + \frac{t}{\mu + t} \max\{-R, -C\} \\
&= \frac{\mu}{\mu + t} (\mu + t - R) + \frac{t}{\mu + t} (-R) \\
&= \mu - R.
\end{aligned}$$

Note that this payoff is independent of the borrower's type and is positive under the assumption that $\mu \geq C$. This implies that all borrower types are willing to borrow from the lender, and there is no screening. Since the sum of the borrower's payoff and the lender's payoff must equal μ , we thus have $V(R) = \mu - U(t, R) = R$.

We now consider the case where $R \geq C$. In this case, the loan rate is sufficiently high so that if the return on the borrower's project is low, the borrower will lose all of his collateral. However, if the return on the borrower's project is high, then the borrower may or may not lose all of his collateral, depending on his type. We use $\theta(R)$ to denote the lowest type whose expected payoff is equal to or greater than zero. Then for any $t \geq \theta(R)$, the type t borrower cannot earn a negative payoff when the realization of the return is high (otherwise, his expected payoff will be negative). So, for any $t \geq \theta(R)$, we have:

$$\begin{aligned}
U(t, R) &= \frac{\mu}{\mu+t} \max\{\mu+t-R, -C\} + \frac{t}{\mu+t} \max\{-R, -C\} \\
&= \frac{\mu}{\mu+t}(\mu+t-R) + \frac{t}{\mu+t}(-C) \\
&= \mu - \frac{\mu}{\mu+t}R - \frac{t}{\mu+t}C \\
&= \mu - R + \frac{t}{\mu+t}(R-C).
\end{aligned}$$

Note that $U(t, R) = 0$ implies that $t = \frac{\mu(R-\mu)}{\mu-C}$. Therefore, $\theta(R) = \max\{x, \frac{\mu(R-\mu)}{\mu-C}\}$. For a given loan rate R , only applicants with types no less than $\theta(R)$ are willing to borrow. Therefore, the expected revenue of the lender is:

$$\begin{aligned}
V(R) &= \mu - E_t(U(t, R)|t \geq \theta(R)) \\
&= \mu - E_t[\mu - R + \frac{t}{\mu+t}(R-C)|t \geq \theta(R)] \\
&= R - (R-C) \int_{\theta(R)}^{\infty} \frac{t}{\mu+t} \frac{g(t)}{1-G(\theta(R))} dt.
\end{aligned}$$

Note that if $R \leq \mu + \frac{x(\mu-C)}{\mu}$, then $\frac{\mu(R-\mu)}{\mu-C} \leq x$, which implies that $\theta(R) = x$. So, if $R \leq \mu + \frac{x(\mu-C)}{\mu}$, then $V(R) = R - (R-C)m = (1-m)R + mC$, where $m = \int_x^{\infty} \frac{t}{\mu+t} g(t) dt = E(\frac{t}{\mu+t}) < 1$.

In summary, we have:

Proposition 1. *The expected revenue of the lender is*

$$V(R) = \begin{cases} R & \text{if } R < C \text{ (no default);} \\ (1-m)R + mC & \text{if } C \leq R \leq \mu + \frac{x(\mu-C)}{\mu} \text{ (some default)} \\ R - (R-C) \int_{\theta(R)}^{\infty} \frac{t}{\mu+t} \frac{g(t)}{1-G(\theta(R))} dt & \text{if } R > \mu + \frac{x(\mu-C)}{\mu} \text{ (adverse selection)} \end{cases}$$

Our next result shows that when unbounded risk is allowed, the lender's expected revenue can be globally hump-shaped.

Theorem 1. *When unbounded risk of borrowers' projects is allowed, the expected revenue of the lender $V(R)$ can be globally hump-shaped.*

Proof:

It is obvious that $V(R)$ is increasing in R for $0 \leq R \leq \mu + \frac{x(\mu - C)}{\mu}$. However, the shape of $V(R)$ for $R > \mu + \frac{x(\mu - C)}{\mu}$ crucially depends on the distribution of borrower types. We next give two examples of distribution of borrower types and show that $V(R)$ is globally hump-shaped in these two cases.

Assume that $\mu = 1.5$ and $C = 0.5$. Figure 1 illustrates $V(R)$ for cases where (i) t follows a Pareto distribution with $\alpha = 1$ and $x = 1$ (i.e., $g(t) = \frac{\alpha x^\alpha}{t^{\alpha+1}} = \frac{1}{t^2}$ for $t \geq 1$), and (ii) t follows a truncated t distribution with degree of freedom = 1 and $x = 0$ (i.e., $g(t) = \frac{2}{\pi(1 + t^2)}$ for $t \geq 0$). In both cases, $V(R)$ is globally hump-shaped.

Formally, it can be shown that in both cases, $V(R)$ will converge to $\frac{\mu + C}{2} = 1$ as R goes to infinity (the proof is in the Appendix). □

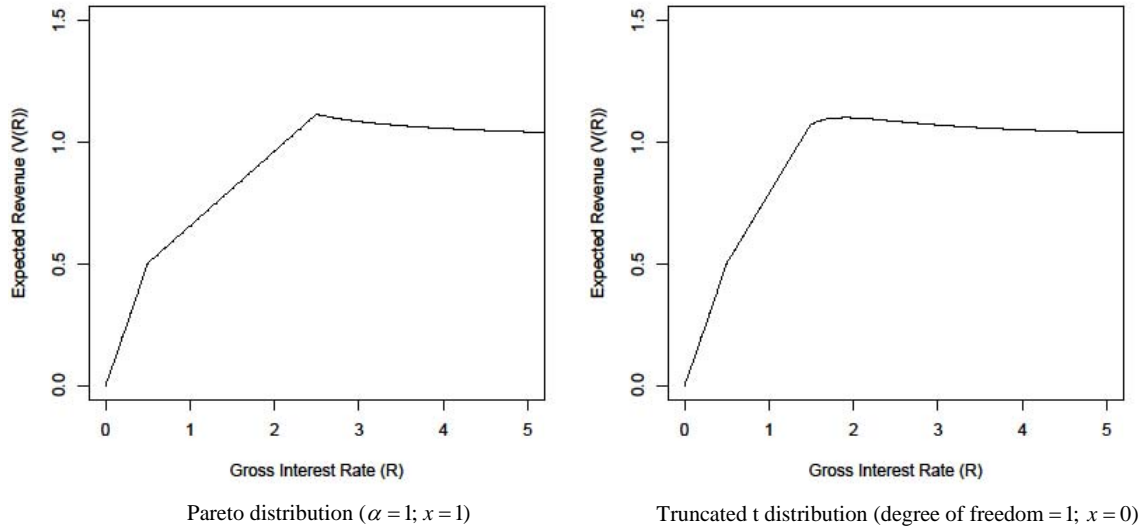


Figure 1

In the proof of Theorem 1, we only provide some examples to show that when unbounded risk of borrowers' projects is allowed, the expected revenue $V(R)$ of the lender can be globally hump-shaped. It should be noted that we are not intended to provide sufficient and necessary conditions of distributions under which $V(R)$ is globally hump-shaped.⁶ Instead, the main

⁶It is a very complicated task to find the sufficient and necessary conditions. However, a general observation is that for heavy-tailed distributions of t , it is likely that $V(R)$ is decreasing for sufficiently large R (and thus $V(R)$ is globally hump-shaped), while for thin-tailed distributions of t , it is likely that $V(R)$ is

purpose of this paper is simply to establish a possibility in which SW model can explain credit rationing correctly, and we indeed find such a possibility (i.e., when borrowers' projects have unbounded risk (or returns)).

3 Conclusion

Arnold and Riley (2009) find that in the credit rationing model of Stiglitz and Weiss (1981), a lender's expected revenue as a function of the loan rate can never be globally hump-shaped, and thus credit rationing is hard to explain using the SW model. A main argument of AR is that "...the SW paper, despite its crucial role in financial economics, is still in need of some clarification" (AR, p. 2012). Our paper answers this question and provides a situation in which the SW model can explain credit rationing correctly. In particular, we find that a lender's expected revenue can be globally hump-shaped when unbounded risk (or equivalently, unbounded returns) about borrowers' projects is allowed. This implies that the SW model can explain credit rationing when banks face a market in which the returns from investment are highly volatile.

References

L. G. Arnold, J. G. Riley, *On the possibility of credit rationing in the Stiglitz-Weiss model*, American Economic Review 99(5) (2009), 2012-2021.

J. E. Stiglitz, A. Weiss, *Credit rationing in markets with imperfect information*, American Economic Review 71(3) (1981), 393-410.

increasing for $R > \mu + \frac{x(\mu - C)}{\mu}$ (and thus $V(R)$ is globally increasing; for example, it can be verified that if t follows (i) a truncated normal distribution of $N(0, 1)$ with $x = 0$ (i.e., $g(t) = \frac{2}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ for $t \geq 0$), or (ii) an exponential distribution with $\lambda = 1$ and $x = 0$ (i.e., $g(t) = \lambda e^{-\lambda t} = e^{-t}$ for $t \geq 0$), then $V(R)$ is globally increasing). However, the above observation is not always true. It can be verified that some heavy-tailed distributions of t can also yield a globally increasing $V(R)$. For example, if t follows a log-normal distribution $\ln N(0, 1)$ (i.e., $g(t) = \frac{1}{t\sqrt{2\pi}}e^{-\frac{(\ln t)^2}{2}}$ for $t > 0$), then $V(R)$ will be globally increasing.

Appendix

Additional Proof of Theorem 1: Suppose t follows a Pareto distribution with $\alpha = 1$ and $x = 1$. That is, $g(t) = \frac{\alpha x^\alpha}{t^{\alpha+1}} = \frac{1}{t^2}$ for $t \geq 1$, and $G(t) = 1 - (\frac{x}{t})^\alpha = 1 - \frac{1}{t}$. We will show that $\lim_{R \rightarrow \infty} V(R) = \frac{\mu + C}{2} = 1$.

Note that when R is sufficiently large, $V(R)$ can be calculated as follows:

$$\begin{aligned}
V(R) &= R - (R - C) \int_{\theta(R)}^{\infty} \frac{t}{\mu + t} \frac{g(t)}{1 - G(\theta(R))} dt \\
&= R - (R - C) \frac{1}{1 - G(\theta(R))} \int_{\theta(R)}^{\infty} \frac{t}{\mu + t} g(t) dt \\
&= R - (R - C) \frac{\mu(R - \mu)}{\mu - C} \int_{\theta(R)}^{\infty} \frac{t}{\mu + t} \frac{1}{t^2} dt \quad (\theta(R) = \frac{\mu(R - \mu)}{\mu - C} \text{ when } R \text{ is large}) \\
&= R - (R - C) \frac{\mu(R - \mu)}{\mu - C} \frac{1}{\mu} \int_{\theta(R)}^{\infty} \left(\frac{1}{t} - \frac{1}{\mu + t} \right) dt \\
&= R - (R - C) \frac{R - \mu}{\mu - C} (\ln t - \ln(\mu + t)) \Big|_{t=\theta(R)}^{\infty} \\
&= R - (R - C) \frac{R - \mu}{\mu - C} \left(-\ln \frac{\theta(R)}{\mu + \theta(R)} \right) \\
&= R + (R - C) \frac{R - \mu}{\mu - C} \ln \frac{R - \mu}{R - C}.
\end{aligned}$$

So, we have:

$$\begin{aligned}
&\lim_{R \rightarrow \infty} V(R) \\
&= \lim_{R \rightarrow \infty} \left[R + (R - C) \frac{R - \mu}{\mu - C} \ln \frac{R - \mu}{R - C} \right] \\
&= \lim_{R \rightarrow \infty} \left[C + (R - C) \left[1 + \frac{R - \mu}{\mu - C} \ln \frac{R - \mu}{R - C} \right] \right] \\
&= C + \lim_{R \rightarrow \infty} \frac{1 + \frac{R - \mu}{\mu - C} \ln \frac{R - \mu}{R - C}}{1/(R - C)} \\
&= C + \lim_{R \rightarrow \infty} \frac{\frac{1}{\mu - C} \left[(R - \mu) \left(\frac{1}{R - \mu} - \frac{1}{R - C} \right) + \ln \frac{R - \mu}{R - C} \right]}{-1/(R - C)^2} \quad (\text{L'Hospital's rule}) \\
&= C + \lim_{R \rightarrow \infty} \frac{\frac{1}{\mu - C} \left[-\frac{\mu - C}{(R - C)^2} + \left(\frac{1}{R - \mu} - \frac{1}{R - C} \right) \right]}{2/(R - C)^3} \quad (\text{L'Hospital's rule})
\end{aligned}$$

$$\begin{aligned} &= C + \lim_{R \rightarrow \infty} \frac{\frac{\mu - C}{(R - \mu)(R - C)^2}}{2/(R - C)^3} \\ &= C + \frac{\mu - C}{2} \\ &= \frac{\mu + C}{2} \\ &= 1. \end{aligned}$$

The proof for the case where t follows a truncated t -distribution is similar and is omitted.