Fair Allocation When Players' Preferences are Unknown

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Abstract: Suppose an arbitrator needs to allocate an asset among two players, whose claims on the asset are incompatible. The allocation procedure is said to be *fair* if the arbitrator awards an outcome that brings the same utility payoff to the two players whenever the two players' claims are symmetric and the allocation set is symmetric. In conjunction with other natural rules, this fairness requirement implies a unique allocation outcome for any claims problem. We propose a mechanism which can be used by the arbitrator to enforce this allocation outcome, even when the players' preferences are *unknown* to the arbitrator.

Keywords: Fair allocation; Claims problem; Implementation.

JEL classification: C72, C78, J52

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Two hold a garment; one claims it all, the other claims half. What is an equitable division of the garment? (Babylonian Talmud, Baba Metzia 2a)

1 Introduction

Suppose there are two players, each of whom claims part of or all of an asset. The claims of both players are legal and verifiable. However, the two players' claims are incompatible with each other, in the sense that the two players' demands cannot be satisfied simultaneously. The question is, how should the asset be divided among the two players, in the event that an arbitrator is called in? Examples of such situations include (i) how should a firm's liquidation value be divided among its creditors when the firm goes bankrupt? and (ii) how should the estate of a person be divided among his heirs, if the heirs to the estate are willed more than the value of the estate?

In the literature, many solutions have been proposed to deal with the *claims problem* above. The most well-know solution is probably the *proportional solution*. It requires that a player obtains an amount that is proportional to the player's claim. In the classic example in the Talmud where player 1 claims all of a good and player 2 claims half of the good, the proportional solution awards 2/3 to player 1 and 1/3 to player 2. Another well-known solution is the *contested garment solution*. It requires that a player be awarded an amount equal to the player's uncontested portion of the asset (i.e., the excess of the total asset over the player's opponent's claim) plus half of the contested portion of the asset (i.e., the excess of the total asset over the sum of the two players' uncontested portions). In the Talmud example, the contested garment solution is (3/4,1/4) for the two players.

A common feature of the proportional solution, the contested garment solution and many other solutions in the literature is that players are awarded different amounts because their claims differ (in other words, the two players are awarded the same amount provided that the two players' claims are the same). This is natural, because players' claims are legal and verifiable, and the arbitrator should respect the players' claims. However, what is missing in the literature is that players usually also differ in their utilities over the allocation of the asset. This paper will introduce players' utilities into claims problems, and we will impose the following requirement on the allocation solution: an arbitrator awards an outcome that brings the two players the same payoff whenever the two players' claims are symmetric and the allocation set[1] is symmetric, where both the claims and the allocation set are measured in terms of players' utilities. We call this requirement the *fairness* requirement. The fairness requirement appears to be a very weak restriction because it provides no prediction for the cases where either the players' claims are not symmetric or the allocation set is not symmetric. We thus need to impose some other requirements on the allocation solution. In this paper, we impose three more requirements and they are Invariance, Pareto Optimality, and Individual Monotonicity.[2]

This paper attempts to answer the following two questions. First, does there exist an allocation solution[3] that satisfies the fairness requirement and the other three requirements? Second, and more importantly, if there exists an allocation solution that satisfies the four requirements, but the arbitrator is ignorant of players' utilities, can we design a mechanism such that the arbitrator can use it to implement the allocation solution we find?

The main results of the paper are as follows. First, we find that there is a unique allocation solution that satisfies all the four requirements. We call this allocation solution the *fair allocation solution*. Second, we find a mechanism, called the *fair allocation procedure*, which implements the fair allocation solution in subgame-perfect equilibrium. That is, for any claims problem, the unique subgame-perfect equilibrium outcome of the fair allocation procedure coincides with the fair allocation solution outcome. Third, as a byproduct of our research, we find a mechanism that implements the Kalai-Smorodinsky bargaining solution.

We assume that a player's claim for himself cannot exceed the total amount of the asset available. Let c_i be the allocation where player i's claim is just satisfied, c_j be the

[1]The allocation set is the set of all possible divisions of the asset between the two players.

[2]All the three requirements are borrowed from the bargaining literature.

[3]Formally, an *allocation solution* is a function that assigns an outcome in the allocation set to each claims problem.

allocation where player j's claim is just satisfied, and a be the final allocation imposed by the arbitrator. We call $\dfrac{u_i(c_i) - u_i(a)}{u_i(c_i) - u_i(c_j)}$ player i's *normalized utility loss*. A player's normalized utility loss measures the player's utility loss from the player's claim to the final allocation (i.e., $u_i(c_i) - u_i(a)$), scaled by the inverse of the player's maximum possible utility loss ($u_i(c_i) - u_i(c_j)$). We show that the fair allocation solution is the unique allocation on the Parero frontier of the allocation set such that the two players' normalized utility losses are identical. Obviously, the players' risk attitudes play an important role in the determination of the fair allocation solution. We find that (i) as a player becomes more risk averse, the player obtains less fair allocation solution payoff, and (ii) when both players are risk neutral, the fair allocation solution coincides with the contested garment solution. This second result is surprising because the principles underlying the fair allocation solution and the contested garment solution are quite different.

The fair allocation procedure we find is a variant of the two-stage alternating-offer game, and is as follows. At the beginning of the game, the arbitrator chooses a player at random, and asks the chosen player to choose a probability $p \in [0, 1]$. The two players then make offers to their opponents sequentially, with the player who was chosen at the beginning of the game being the first proposing player. At any stage, the responding player can either accept the proposing player's offer or reject it. In addition, the arbitrator can end the game whenever a responding player has just rejected a proposing player's offer and the responding player requests to do so. We assume that if the arbitrator is requested by a responding player, say player i, to end the game, then the game ends with the lottery where player i's claim is chosen with probability p and player j's claim is chosen with probability $1 - p$ (i.e., $p \cdot c_i + (1 - p) \cdot c_j$ is the outcome).

It can be shown that the unique subgame-perfect equilibrium outcome of the fair allocation procedure coincides with the fair allocation solution outcome for any claims problem. Moreover, the fair allocation procedure is independent of players' utilities and the arbitrator does not require any knowledge of players' utilities in order to enforce this procedure. Our research thus provides a *non-cooperative* foundation for the fair allocation solution.

A byproduct of our research is that, based on the fair allocation procedure, we can design a mechanism that implements the *Kalai-Smorodinsky solution*. The mechanism is called the *strategic claim with fair allocation procedure*, and is described below. At stage 1 (the pre-arbitration stage), two players *strategically* choose their claims for arbitration. At stage 2 (the arbitration stage), the arbitrator uses the *fair allocation procedure* to determine the outcome. One can show that the two players will submit extreme claims in equilibrium and that the corresponding fair allocation outcome coincides with the Kalai-Smorodinsky solution outcome.

Related literature. Our paper is mostly related to the literature about *claims problems* (see Thomson (2003) for a survey of the literature). The claims problem defined in this paper is more general than the claims problem in the literature because our claims problem takes players' utilities into consideration. Chun and Thomson (1992) also considered a situation where players' utilities are introduced into a "claims problem". The difference between the two papers is that the problem considered in our paper is a *claims problem* in which the arbitrator chooses a final allocation based on the players' claims and utilities. By contrast, the problem considered in Chun and Thomson (1992) is essentially a *bargaining problem*, where the two players' claims are only "reference points" for the two players to bargain with each other (and if the two players cannot reach agreement, then the disagreement point will be enforced).

Our paper is also related to Rong (2012). The fair allocation solution is actually the same as the *symmetric arbitration solution* proposed in Rong (2012). The differences between the two papers are as follows. First, the characterization of the fair allocation solution in this paper is a strengthening of the characterization of the symmetric arbitration solution in Rong (2012).[4] Second, and most importantly, Rong (2012) mainly focuses on the *characterization*

[4]Rong (2012) mainly focuses on using a simpler but stronger version of the fairness requirement to characterize the symmetric arbitration solution. This stronger requirement requires that the allocation payoff be the same for the two players whenever the two players claims are symmetric. Rong (2012) finds that the symmetric arbitration solution is the unique allocation solution that satisfies this stronger fairness

of the symmetric arbitration solution, while this paper mainly focuses on the *properties* and the *implementation* of the fair allocation solution.

Our paper is also closely related to the literature about the Nash program. More particularly, the strategic claim with fair allocation procedure proposed in this paper is a mechanism that *implements* the Kalai-Smorodinsky solution. Moulin (1984) also designed a mechanism, known as the *auctioning fractions of dictatorship mechanism*,[5] which implements the Kalai-Smorodinsky solution. Our mechanism differs from Moulin (1984) in the sense that the auctioning fractions of dictatorship mechanism has an auction framework, while our mechanism has a simultaneous-offer and alternating-offer framework. However, both mechanisms have the same flavor, in the sense that the implementation of the Kalai-Smorodinsky solution in both mechanisms relies on the players' extreme threats.[6] This feature is also known as "action at a distance" in the Kalai-Smorodinsky solution (Serrano (2005)).

This paper is organized as follows. Section 2 presents the basic elements of the model. Section 3 studies the properties of the fair allocation solution. Section 4 discusses the mechanism that implements the fair allocation solution. Section 5 discusses the mechanism that implements the Kalai-Smorodinsky solution. Concluding remarks are offered in Section 6.

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requirement and two other rules (Invariance and Pareto Optimality). Rong (2012) also provides an axiomatic characterization of the symmetric arbitration solution using the same fairness requirement as this paper, and shows that the symmetric arbitration solution is the unique solution that satisfies the fairness requirement and three other rules (Invariance, Strong Monotonicity and Pareto Optimality). Our axiomatic characterization of the fair allocation solution is a strengthening of this second axiomatic characterization in Rong (2012), because the individual monotonicity axiom that we use is weaker than the strong monotonicity axiom used in Rong (2012).

[5]In the auctioning fractions of dictatorship mechanism, two players simultaneously make bids between 0 and 1. The bidder with the highest bid is the winning bidder. The winning bidder makes an offer to the non-winning bidder. If the offer is accepted, then the offer is implemented. If the offer is rejected, then the non-winning bidder makes a "take it or leave it" offer to the winning bidder with probability p (where p is the bid of the winning bidder), and the disagreement point is implemented with probability $1 - p$.

[6]However, notice that there is a difference between the two mechanisms. In the auctioning fractions of dictatorship mechanism, it is a design of the mechanism for players to make extreme threats upon the rejection of an offer (see also footnote 5). By contrast, in our mechanism, it is an equilibrium behavior for players to make extreme claims.

2 Preliminaries

2.1 The model

Suppose there is a perfectly divisible asset to be divided among two players. Let $X = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y \leq E\}$ be the set of all possible *deterministic* allocations of the asset among the two players, where x represents player 1's allocation, y represents player 2's allocation and E is the total amount of the asset available. Let $P(X)$ be the Pareto frontier of X, i.e., $P(X) = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y = E\}$. We allow randomization in the allocation. That is, an allocation can be a lottery over deterministic allocations in X. We use $\Delta(X)$ to denote the set of all lotteries over X, and call $\Delta(X)$ the *allocation set*. Let $u_i : \Delta(X) \to R$ denote player i's *expected utility* function. We assume that u_i only depends on player i's own allocation, and when restricted in X, u_i is continuous and strongly monotonic in player i's own allocation. Let $S(\Delta(X), u_1, u_2) = \{(u_1(a), u_2(a)) : a \in \Delta(X)\}$ be the image of $\Delta(X)$ under the utility functions (u_1, u_2). $S(\Delta(X), u_1, u_2)$ is thus the allocation set measured in terms of players' utilities.[7] For simplicity, we write $S(\Delta(X), u_1, u_2)$ as S whenever there is no confusion. The Pareto frontier of S is denoted by $P(S) = \{s \in S | s' >> s \Rightarrow s' \notin S\}$. Notice that S is a convex set because $\Delta(X)$ is a set of lotteries and u_1 and u_2 represent expected utilities. We use d to denote the allocation $(0, 0)$. A typical allocation set is illustrated in Figure 1.

We use $c_1 = (c_{11}, c_{12})$ to denote player 1's *claim*, and $c_2 = (c_{21}, c_{22})$ to denote player 2's *claim*. We require that $c_1 \in P(X)$ and $c_2 \in P(X)$.[8] In our definition, a player's claim is an allocation plan, which not only includes the player's demand (c_{ii}) for himself, but also includes the player's suggested allocation (c_{ij}) for his opponent.[9] Although we define

[7]In the remainder of the paper, the "allocation set" either refers to the set $\Delta(X)$, or the set $S(\Delta(X), u_1, u_2)$. The exact meaning is usually clear in the context in which it appears.

[8]This assumption is imposed for simplicity. The main results of the paper can be extended to the general case where $c_1 \in \Delta(X)$ and $c_2 \in \Delta(X)$.

[9]The assumption that $c_i \in P(X)$ implies that a player's demand for himself c_{ii} does not exceed E. In this paper, c_{ii} should be understood as player i's *effective* demand for himself. That is, if a player demands more than E for himself, then his demand will be truncated by E such that the player's effective demand for himself never exceeds E.

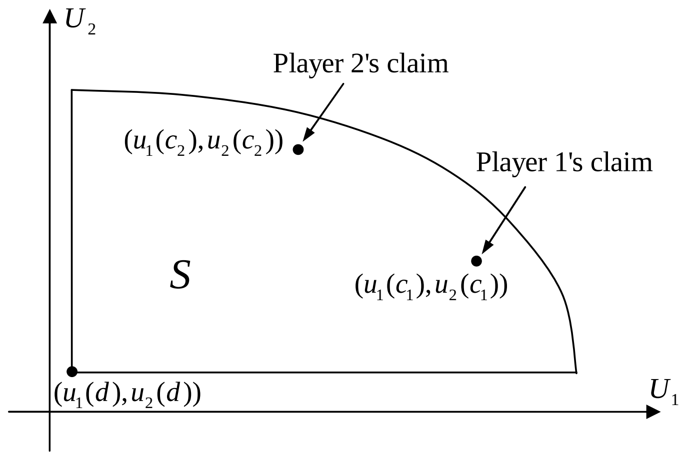


Figure 1: Allocation set and incompatible claims.

a player's claim as a two-dimensional vector, rather than a one-dimensional value, which is usually the case in the literature, the meaning of our definition is essentially the same as the definition in the literature because we require a player's claim be on the Pareto frontier of X. That is, for player i, as long as the player's claim for himself is given, then his suggested allocation for his opponent is also determined.

We assume that the two players' claims are *legal* and *verifiable*, so that a player cannot strategically report his claim. In addition, we assume that the two players' claims are *incompatible*, meaning that there is no allocation $a \in \Delta(X)$ such that $(u_1(c_1), u_2(c_2)) = (u_1(a), u_2(a))$ (see Figure 1). Notice that although we require the two players' (physical) claims be on the Pareto frontier of $P(X)$, the two players' utility claims may not lie on the Pareto frontier of S.[10]

We call (c_1, c_2, E, u_1, u_2) a *claims problem*, and use $\mathcal{C}$ to denote the set of all claims problems that satisfy the assumptions mentioned above. An *allocation solution* is a mapping $g : \mathcal{C} \to \Delta(X)$ that associates a lottery in $\Delta(X)$ with each claims problem. A *mechanism* (or, *game form*) Γ is a triple $(\Sigma_1, \Sigma_2; h)$, where Σ_i is the strategy set of player i and Σ_i is independent of the players' utilities, and $h : \Sigma_1 \times \Sigma_2 \to \Delta(X)$ is an outcome function

[10]This occurs, for example, when both players are risk seeking (i.e., for $i = 1, 2$, u_i is a convex function in player i's own allocation when the function is restricted in the domain X). In this case, the image of $P(X)$ under the two players' utilities is a convex curve in R^2, while the Pareto frontier of S is a line which lies above the image of $P(X)$ under the two players' utilities.

(Dagan and Serrano (1998); Trockel (2002)). A mechanism Γ is said to *implement* an allocation solution g in subgame-perfect equilibrium if for any claims problem $C \in \mathcal{C}$, $g(C)$ is the unique pure strategy subgame-perfect equilibrium outcome of the game (Γ, C).

2.2 Fairness requirement on allocation solution

We say an allocation solution is *fair* if it assigns the same payoff to the two players whenever the two players' claims (measured in terms of utility level) are symmetric and the allocation set (also measured in terms of utility level) is symmetric. That is, an allocation solution g is fair if it satisfies the following rule.

- **Fairness** (F): For any claims problem $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$ where $u_1(c_1) = u_2(c_2)$, $u_2(c_1) = u_1(c_2)$ and S is symmetric, we have $u_1(g(c_1, c_2, E, u_1, u_2)) = u_2(g(c_1, c_2, E, u_1, u_2))$.

It is obvious that the fairness requirement is insufficient to determine any allocation solution. We thus impose three additional natural rules on the allocation solution. These rules are Invariance, Pareto Optimality, and Individual Monotonicity. Invariance requires that the (physical) allocation solution outcome be the same under positive affine transformations of players' utilities. The idea behind this rule is that the allocation outcome should only depend on players' preferences and not on the precise utility representations of players. Since under the expected utility assumption, a player's utility is unique up to positive affine transformations, the (physical) allocation outcome should be invariant to any positive affine transformation of players' utilities. Pareto optimality requires that the allocation solution outcome be on the Pareto frontier of the allocation set for any claims problem. Individual Monotonicity requires that a player obtain a greater allocation payoff when the part of the Pareto frontier lying between the two players' claims expands from the viewpoint of the player, with the two players' claims being fixed.[11]

[11]Notice that Individual Monotonicity is a weaker requirement than Strong Monotonicity defined in Rong (2012), which requires that a player obtain a greater allocation payoff as the entire allocation set expands

Let g be an allocation solution. The rules Invariance, Pareto Optimality, and Individual Monotonicity are defined as follows.

- **Invariance** (INV): If $T : R^2 \to R^2$ represents a positive affine transformation, i.e., $T(x, y) = (r_1 x + s_1, r_2 y + s_2)$ for some positive constant r_i and some constant s_i, then we have $g(c_1, c_2, E, T(u_1, u_2)) = g(c_1, c_2, E, u_1, u_2)$ for any $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$.

- **Pareto Optimality** (PO): For any claims problem $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$, we have $(u_1(g(c_1, c_2, E, u_1, u_2)), u_2(g(c_1, c_2, E, u_1, u_2))) \in P(S)$.

- **Individual Monotonicity** (IM): For any two claims problems $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$ and $(c'_1, c'_2, E', u'_1, u'_2) \in \mathcal{C}$, if $u_1(c_1) = u_1(c'_1)$, $u_2(c_1) = u_2(c'_1)$, $u_1(c_2) = u_1(c'_2)$, $u_2(c_2) = u_2(c'_2)$, and $f_i(x_j, E, u_1, u_2) \leq f_i(x_j, E', u'_1, u'_2)$ for any $x_j \in [u_j(c_i), u_j(c_j)]$, then $u_i(g(c_1, c_2, E, u_1, u_2)) \leq u_i(g(c'_1, c'_2, E', u'_1, u'_2))$ (where $f_i(x_j, E, u_1, u_2)$ is the maximal possible payoff for i in S given that player j's payoff is x_j).

We obtain the following result.

Theorem 1. *There is a unique allocation solution,*[12] *called the* ***fair allocation solution****, which satisfies F, INV, PO, and IM. Let $a^* \in \Delta(X)$ be the fair allocation solution outcome to a given claims problem $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$, then $(u_1(a^*), u_2(a^*))$ must be the unique point on the Pareto frontier of the allocation set S such that $\dfrac{u_1(c_1) - u_1(a^*)}{u_1(c_1) - u_1(c_2)} = \dfrac{u_2(c_2) - u_2(a^*)}{u_2(c_2) - u_2(c_1)}$.*

Proof: see the appendix. □

The ratio $\dfrac{u_i(c_i) - u_i(a^*)}{u_i(c_i) - u_i(c_j)}$ represents player i's *normalized utility loss*, since $u_i(c_i) - u_i(a^*)$ is player i's utility loss from his claim point c_i to the allocation outcome a^* and $u_i(c_i) - u_i(c_j)$ is the maximum possible utility loss that player i can have (because the worst case for player

(Individual Monotonicity only requires that "part of" the allocation set expands while the other part of the allocation set might actually shrink).

[12]The solution in unique up to payoff. That is, any two fair allocation solutions must yield the same payoffs for the two players for any given claims problem.

i is to have player j's claim as the allocation outcome). The fair allocation solution thus simply requires that the two players' normalized utility losses be identical.

The fair allocation solution is illustrated in Figure 2. The figure shows that the fair allocation solution is also the intersection point of the Pareto frontier of the allocation set and the line joining the component-wise minimum and the component-wise maximum of the two players' claims.

The idea of the proof of Theorem 1 is derived from Kalai and Smorodinsky (1975) and Rong (2012). For completeness, we provide a sketch of the proof in the appendix. The intuition of the proof is as follows. For any asymmetric claims problem, we can always find a positive affine transformation to transform it into a new claims problem, in which the two players' claims in the new problem are symmetric. The allocation set of this new claims problem is generally still asymmetric. However, we can always find a symmetric allocation set, which is contained in the above asymmetric allocation set, and determine the allocation solution for this smaller claims problem using the fairness requirement and Pareto Optimality. Finally, using Individual Monotonicity and Invariance allows us to determine the allocation solution to the original problem.

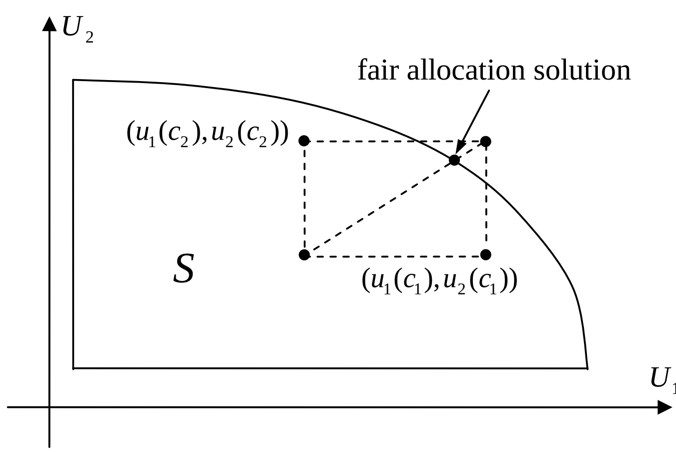


Figure 2: The fair allocation solution.

The following example illustrates how the fair allocation solution outcome is determined for the simple case where there is a unit of asset with player 1 claiming all of the asset and player 2 claiming half of the asset. We allow players' utilities to be either risk-neutral or risk-averse.

Example 1. *Suppose the set of deterministic allocations* $X = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y \leq 1\}$. *Suppose player 1's claim is* $(1, 0)$, *and player 2's claim is* $(\frac{1}{2}, \frac{1}{2})$. *Obviously, the two players' claims are incompatible.*

If the two players' utilities on X *are such that* $u_1(x, y) = x$ *and* $u_2(x, y) = y$ *for any* $(x, y) \in X$ *(i.e., both players are risk neutral), then it can be shown that the fair allocation solution outcome is* $(\frac{3}{4}, \frac{1}{4})$. *The normalized utility losses of both players are 1/2.*

If the two players' utilities on X *are such that* $u_1(x, y) = x$ *and* $u_2(x, y) = \sqrt{y}$ *for any* $(x, y) \in X$, *then the fair allocation solution outcome is* $(0.809, 0.191)$. *The normalized utility losses of both players are 0.382.*

If the two players' utilities on X *are such that* $u_1(x, y) = \sqrt{x}$ *and* $u_2(x, y) = y$ *for any* $(x, y) \in X$, *then the fair allocation solution outcome is* $(0.739, 0.261)$. *The normalized utility losses of both players are 0.478.*

3 Properties of the fair allocation solution

3.1 Risk sensitivity in fair allocation

Example 1 shows that as a player's preference changes from risk neutral preference to risk averse preference, the player's fair allocation solution outcome becomes worse for the player. In this subsection, we show that this result is true for a very general class of claims problems. In particular, we show that as long as the claims problems are deterministic, then risk aversion is disadvantageous for a player under the fair allocation solution.

We say that a claims problem (c_1, c_2, E, u_1, u_2) is *deterministic* if the fair allocation solution outcome to the problem is a deterministic allocation. The set of deterministic claims problems are very general. It includes, for example, problems where (i) the Pareto frontier of the allocation set S is strictly concave, or (ii) there is no randomization, i.e., the allocation set is X, rather than $\Delta(X)$.

Let u_i and v_i be two utility functions defined on $\Delta(X)$. We call v_i is *more risk averse* than u_i if there exists a concave and strictly increasing function k such that $v_i(x) = k(u_i(x))$ for any $x \in X$.

We use g^* to denote the fair allocation solution. We obtain the following result.

Proposition 1. *Suppose there are two claims problems* (c_1, c_2, E, u_1, u_2) *and* (c_1, c_2, E, u_1, v_2)*, where both problems are deterministic and* v_2 *is more risk averse than* u_2*. Then we have* $u_2(a^*) \geq u_2(a^{*\prime})$*, where* $a^* = g^*(c_1, c_2, E, u_1, u_2)$*,* $a^{*\prime} = g^*(c_1, c_2, E, u_1, v_2)$*.*

Proof: see the appendix. □

In order to illustrate the intuition of the proof of Proposition 1, let us assume that the two players' claims are on the Pareto frontier (refer to Figure 3).[13] It is without loss of generality to assume that $v_2(c_1) = u_2(c_1)$ and $v_2(c_2) = u_2(c_2)$ (otherwise, we can make a positive affine transformation on v_2 such that the two equalities hold). Given that v_2 is more risk averse than u_2, the Pareto frontier of the allocation set under (u_1, v_2) is more "bowed-out" than the Pareto frontier of the allocation set under (u_1, u_2). This implies that player 2 obtains a smaller payoff when player 2's utility is v_2 than when player 2's utility is u_2 (i.e., $v_2(a^{*\prime}) < v_2(a^*)$ or $u_2(a^{*\prime}) < u_2(a^*)$ where a^* is the fair allocation solution outcome when player 2's utility is u_2 and $a^{*\prime}$ is the fair allocation solution outcome when player 2's utility is v_2).

The intuition of Proposition 1 can be explained as follows. The fair allocation solution requires that the two players' normalized utility losses be identical. If player 2 becomes more risk averse, then player 2's normalized utility loss, when evaluated at the original fair allocation outcome, becomes smaller (i.e., $\frac{v_2(c_2) - v_2(a^*)}{v_2(c_2) - v_2(c_1)} \leq \frac{u_2(c_2) - u_2(a^*)}{u_2(c_2) - u_2(c_1)}$). In order to

[13]This assumption is made for the purpose of exhibition, and it is not required in the formal proof of Proposition 1.

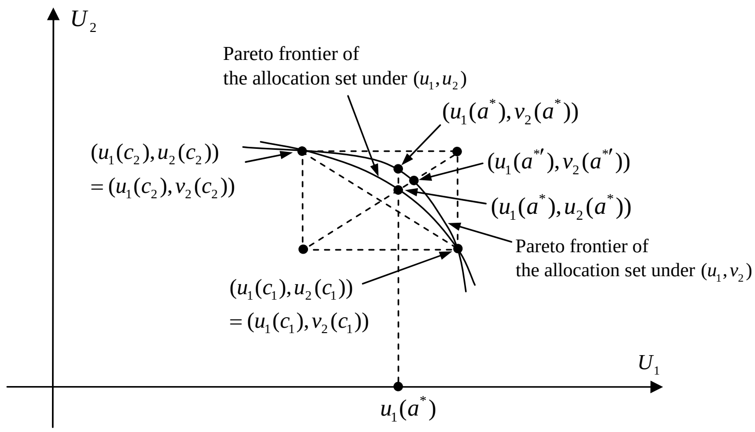


Figure 3: Comparison of two claims problems.

re-balance the two players' normalized utility losses, player 2 thus needs to give up more. As a result, player 2 obtains less payoff when his utility becomes more risk averse.

3.2 Relationship with the contested garment solution

This subsection studies the relationship between our fair allocation solution and the contested garment solution. In particular, we find that the fair allocation solution coincides with the contested garment solution if both players have *risk-neutral* preferences.

The *contested garment solution* outcome for the claims problem (c_1, c_2, E, u_1, u_2) is $(\frac{E-(E-c_{11})-(E-c_{22})}{2} + E - c_{22}, \frac{E-(E-c_{11})-(E-c_{22})}{2} + E - c_{11})$ (see, e.g., Aumann and Maschler (1985)).[14] The idea behind the contested garment solution is that each player can obtain at least his uncontested portion (i.e., $E - c_{jj}$ for player i). In addition, each player obtains half of the contested portion (i.e., the excess of E over the sum of the two players' uncontested portions).

Notice that $(\frac{E-(E-c_{11})-(E-c_{22})}{2} + E - c_{22}, \frac{E-(E-c_{11})-(E-c_{22})}{2} + E - c_{11}) = (c_{11} - \frac{c_{11}+c_{22}-E}{2}, c_{22} - \frac{c_{11}+c_{22}-E}{2})$. So, for the two player case, the contested garment solution of (c_1, c_2, E) requires equal (physical) losses for the two players. When players

[14]Recall that we assume that a player's demand for himself cannot exceed E, i.e., $c_{ii} \leq E$ for $i = 1, 2$. If instead, a player's demand for himself exceeds E, then we need to first truncate the player's demand by E, and the result below (Proposition 2) still holds.

preferences are risk neutral, equal (physical) losses implies equal normalized utility losses. As a result, the contested garment solution coincides with the fair allocation solution when both players are risk neutral. That is, we have:

Proposition 2. *(Relationship with Contested Garment Solution) If u_1 and u_2 represent risk-neutral preferences, then the contested garment solution payoff to any given claims problem (c_1, c_2, E, u_1, u_2) is the same as the fair allocation solution payoff to the problem.*

4 Fair allocation procedure

In this section, we assume that the arbitrator does not know the players' preferences.[15] We propose an arbitration procedure known as the *fair allocation procedure* to implement the fair allocation solution. The difficulty of finding a mechanism to implement an allocation solution is that such a mechanism should not rely on players' preferences, so the mechanism can be used by the arbitrator even when the arbitrator is ignorant of players' preferences.

We use $p \cdot a_1 + (1-p) \cdot a_2$ to denote the lottery that assigns probability p to the allocation $a_1 \in \Delta(X)$ and assigns probability $1-p$ to the allocation $a_2 \in \Delta(X)$. The *fair allocation procedure* is defined as follows.

- Stage 0:
  - The arbitrator chooses a player at random. The chosen player, say player i, chooses a probability $p \in [0, 1]$.
- Stage 1:
  - Player i makes an offer to player j. This offer is denoted by $c_i' \in \Delta(X)$.
  - Player j chooses whether to accept the offer or reject it. If player j chooses to accept the offer, then c_i' is the outcome. Otherwise, player j can choose one of the

[15] We still assume that the players know each other's preferences. A typical example that fits this assumption is the King Solomon's dilemma, in which the two women knew each other's preference, but the King Solomon did not know the two women's preferences.

following two options: (i) let the arbitrator end the game with $p \cdot c_j + (1-p) \cdot c_i$, and (ii) let the arbitrator move the game to the next stage.

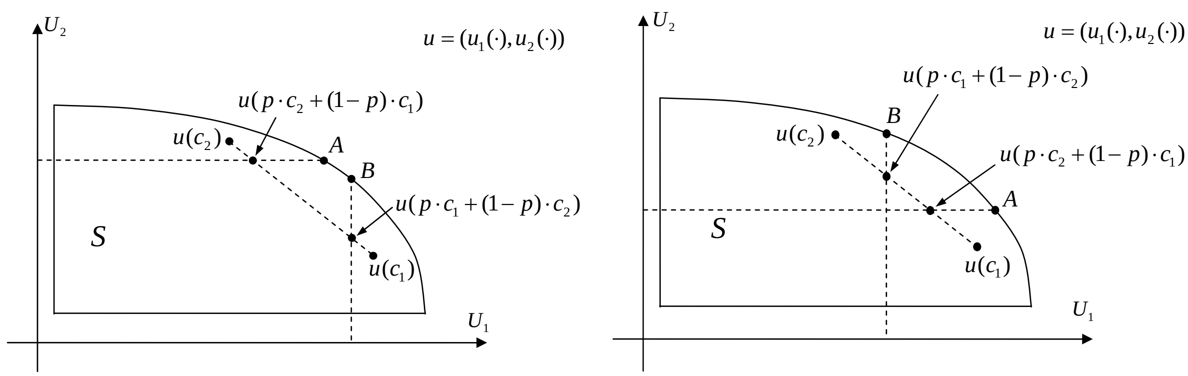
- Stage 2:
  - Player j makes an offer to player i. This offer is denoted by $c'_j \in \Delta(X)$.
  - Player i chooses whether to accept the offer or reject it. If player i chooses to accept the offer, then c'_j is the outcome. Otherwise, player i can either let the arbitrator end the game with $p \cdot c_i + (1-p) \cdot c_j$, or let the arbitrator move the game to the next stage.
- Stage 3:
  - $(0, 0)$ is the outcome.

The fair allocation procedure is a variant of a two-stage alternating-offer game. A novel feature of the fair allocation procedure is that whenever a player rejects his opponent's offer, the player can ask the arbitrator to end the game. In particular, we assume that if player i requests that the arbitrator end the game, then the game ends with $p \cdot c_i + (1-p) \cdot c_j$. This additional option that is available to a responding player can be regarded as the player's *outside option*. We obtain the following result.

Theorem 2. *For any given claims problem $C \in \mathcal{C}$, the unique SPE payoff of the fair allocation procedure coincides with the fair allocation solution payoff.*

Proof: see the appendix. □

Assuming that player 1 is chosen at stage 0, the intuition of Theorem 2 can be explained as follows. The key to the equilibrium analysis of the fair allocation procedure is to determine the probability p that player 1 will choose at stage 0. Obviously, if p is large, then both players have good outside options. If p is small, then both players have bad outside options. In equilibrium, p cannot be too large, because otherwise, player 1 will have to make a very favorable offer to player 2 because player 2 has a very good outside option. On the other hand, p cannot be too small, because otherwise, player 1 still must make a very favorable offer to player 2. This is because if player 1's offer is not sufficiently favorable to player 2, then player 2 will reject player 1's offer and make a very unfavorable offer to player 1, and player 1 will have to accept such an offer because player 1 does not have a good outside option. It turns out that in equilibrium, p must be such that player 2 is indifferent between his two options upon rejection: allowing the arbitrator end the game (i.e., ending the game with player 2's outside option) and moving the game to the next stage (i.e., ending the game with an allocation that is on the Pareto frontier and gives player 1 the same payoff as player 1's outside option).



The case where p is large

The case where p is small

Figure 4

Figure 4 and Figure 5 further illustrate the observations above. The left-hand figure in Figure 4 illustrates the case where the probability p is large. In this case, each player's outside option is close to the player's claim, and the equilibrium of the game is that player 1 makes an offer at A, which player 2 accepts. The right-hand figure in Figure 4 illustrates the case where the probability p is small. In this case, each player's outside option is distant from the player's claim, and the equilibrium is that player 1 makes an offer at B, which player 2 accepts. Obviously, player 1's payoff is maximized when the probability p is such that A and B coincide with each other (see Figure 5). In this case, the equilibrium offer (a^*) that player 1 makes is exactly the same as the fair allocation solution outcome.

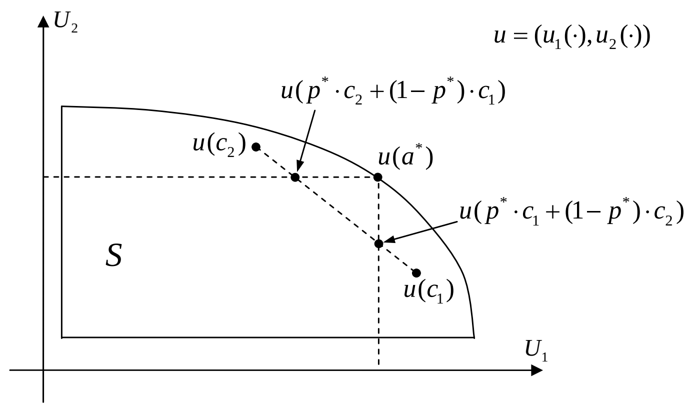


Figure 5

We can modify the fair allocation procedure such that the two players simultaneously report probabilities at the beginning of the game. The player with the higher probability becomes the winner, and the game proceeds with the winner as the first proposer. The rest of the game is the same as the fair allocation procedure. It can be shown that this new game also yields the same equilibrium outcome as the fair allocation procedure, and it shares the flavor of the game in Moulin (1984) in the sense that both games have an auction stage at the beginning of the game.

5 Implementation of the Kalai-Smorodinsky solution

Up to this point, we have assumed that both players' claims are legal and verifiable, and that players cannot voluntarily "choose" their claims. In this section, we assume that players' claims are *not* legal and verifiable and thus players can strategically report their claims. Alternatively, we can understand the situation where players can strategically choose their claims as a bargaining situation.

We call $(d, \Delta(X), u_1, u_2)$ a *bargaining problem*, where $d = (0, 0)$ is the *disagreement point*, $\Delta(X)$ is the allocation set, and u_1 and u_2 are the two players' expected utilities. We use b_i to denote player i's maximal possible utility level from the allocation set S. The *Kalai-Smorodinsky solution* to the bargaining problem $(d, \Delta(X), u_1, u_2)$ is determined by the intersection point of the Pareto frontier of S and the line joining $(u_1(d), u_2(d))$ and (b_1, b_2).

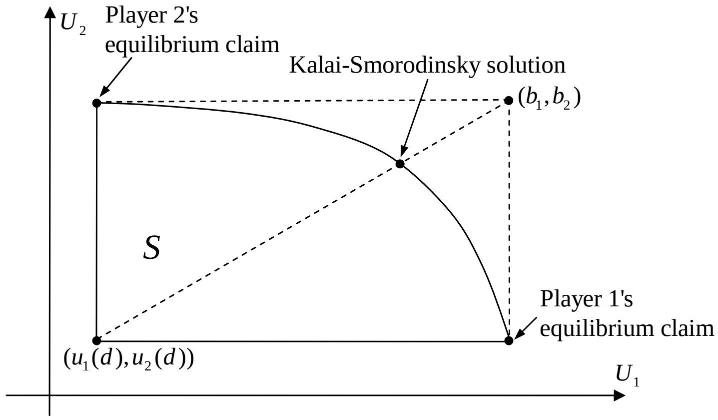


Figure 6: Implementation of the Kalai-Smorodinsky Solution.

Notice that a property of our fair allocation solution is that as a player claims more for himself, then the player's fair allocation solution payoff increases. As a result, if both players are allowed to strategically submit their claims to the arbitrator and both players know that the arbitrator will use the fair allocation solution to determine the outcome, then each player has an incentive to submit a claim that is as extreme as possible. Eventually, both players will submit extreme claims, and the corresponding arbitration outcome coincides with the *Kalai-Smorodinsky solution* outcome (refer to Figure 6).[16] Combining this observation and our fair allocation procedure, we then obtain a mechanism, called the *strategic claim with fair allocation procedure*, which *implements* the Kalai-Smorodinsky solution. Formally, we define the *strategic claim with fair allocation procedure* as follows:

- *Pre-arbitration stage:* Player 1 and player 2 strategically report their claims, $c_1 \in \Delta(X)$ and $c_2 \in \Delta(X)$, respectively, to the arbitrator.
- *Arbitration stage:* The fair allocation procedure is utilized to determine the final outcome.

We obtain the following result.

[16] See also Rong (2012) for this observation.

Theorem 3. *For any given bargaining problem* $(d, \Delta(X), u_1, u_2)$*, the unique SPE outcome of the strategic claim with fair allocation procedure coincides with the Kalai-Smorodinsky solution outcome.*

Proof: see the appendix. □

Although we assume that $X = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y \leq E\}$ and $d = (0, 0)$ throughout the paper, Theorem 3 can be easily extended to the case where X is any compact set such that (i) $S(\Delta(X), u_1, u_2)$ is *d-comprehensive* and (ii) $d \in X$ is the least preferred allocation for both players. A set S is d-comprehensive if for any $(x, y) \in S$, if there exists a (x', y') such that $(u_1(d), u_2(d)) \leq (x', y') \leq (x, y)$, then we must have $(x', y') \in S$.

6 Concluding remarks

This paper considers the scenario where two players are unable to reach agreement with each other, and the two players submit their claims to an arbitrator for arbitration. Both players' claims are legal and verifiable. We propose that a fairness requirement be imposed on the arbitration procedure, which requires that the arbitrator award an outcome that brings the same payoff to the two players whenever the two players' claims are symmetric and the allocation set is symmetric, where both the players' claims and the allocation set are measured in terms of players' utilities. We show that together with other natural rules, this fairness requirement implies a unique allocation outcome, the fair allocation solution outcome, for any claims problem. We then propose an arbitration procedure, called the fair allocation procedure, which implements the fair allocation solution. Based on the fair allocation procedure, we design a mechanism that implements the Kalai-Smorodinsky solution.

The fair allocation procedure involves randomness in the game form. This is not surprising, given that the equilibrium concept (subgame-perfect equilibrium) we use is *ordinally invariant*, while the solution (the fair allocation solution) we want to implement is *scale* *invariant*. In order to avoid this conflict between the equilibrium concept and the solutions, randomness in the game form is needed (Dagan and Serrano (1998)).

Finally, it is worth noticing that our implementation result for the Kalai-Smorodinsky solution differs from the various support results in the literature regarding the Kalai-Smorodinsky solution (e.g., Trockel (1999), Haake (2000), Anbarci and Boyd (2011)). In order to support an axiomatic solution, one only needs to design a strategic game (instead of a mechanism) whose equilibrium outcome coincides with the axiomatic solution outcome. A strategic game might depend on players' utilities, and thus the game itself might change as the players' utilities change. In comparison with the support results, designing a mechanism to implement an axiomatic solution is a more challenging work.

Appendix

Proof of Theorem 1:

We use γ to denote the fair allocation solution. It is easy to verify that γ satisfies F, INV, PO, and IM. Assume that there is another allocation solution μ that satisfies the four axioms. In order to prove that γ is the unique allocation solution that satisfies all the four axioms, it is sufficient to show that $\mu(c_1, c_2, E, u_1, u_2) = \gamma(c_1, c_2, E, u_1, u_2)$ for any $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$.

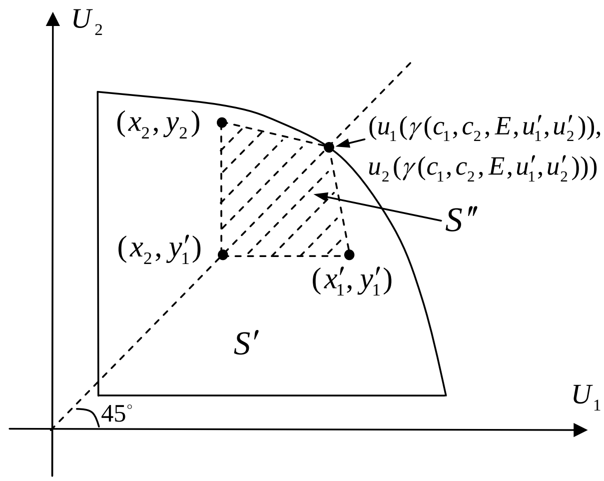


Figure 7: The claims problem $(c_1, c_2, E, u_1', u_2')$ in the utility payoff space.

Suppose that the claims problem $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$ is given. Let $(x_1, y_1) = (u_1(c_1), u_2(c_1))$ and $(x_2, y_2) = (u_1(c_2), u_2(c_2))$. The incompatibility of c_1 and c_2 implies that $x_1 > x_2$ and $y_2 > y_1$. It is without loss of generality to assume that $y_2 > x_2$.[17] Let $T^*(x, y) = (\frac{y_2 - x_2}{x_1 - x_2}x + \frac{x_2(x_1 - y_2)}{x_1 - x_2}, \frac{x_2 - y_2}{y_1 - y_2}y + \frac{y_2(y_1 - x_2)}{y_1 - y_2})$. Then $T^*(x, y)$ is a positive affine transformation. Let $(u_1'(\cdot), u_2'(\cdot)) = (T^*(u_1(\cdot), u_2(\cdot)))$. Let $(x_1', y_1') = (u_1'(c_1), u_2'(c_1))$ and $(x_2', y_2') = (u_1'(c_2), u_2'(c_2))$. It can be verified that $x_2' = x_2$, $y_2' = y_2$, $x_1' = y_2$ and $y_1' = x_2$. That is, for the new claims problem $(c_1, c_2, E, u_1', u_2')$, the two players' claims (measured in terms of utility level) are symmetric. Figure 7 illustrates this new claims problem in the utility payoff space.

[17] If $y_2 \leq x_2$, then we can use the transformation $T(x, y) = (x, y + c)$ (where c is sufficiently large) to transform the claims problem to a new problem, which has the property $y_2' > x_2'$ (where $y_2' = y_2 + c$ and $x_2' = x_2$).

Let S'' be the convex hull of points (x'_1, y'_1), (x_2, y_2), (x_2, y'_1) and $(u_1(\gamma(c_1, c_2, E, u'_1, u'_2)), u_2(\gamma(c_1, c_2, E, u'_1, u'_2)))$. Notice that S'' is a symmetric set. We can find a claims problem $(c''_1, c''_2, E'', u''_1, u''_2)$ such that the allocation set (measured in terms of utility level) of the claims problem is S'' and the two players' claims (measured in terms of utility level) are (x'_1, y'_1) and (x_2, y_2) respectively. So, we must have $\mu(c''_1, c''_2, E'', u''_1, u''_2) = \gamma(c_1, c_2, E, u'_1, u'_2)$ by axiom F and axiom PO. In addition, since $f_i(x_j, E'', u''_1, u''_2) \leq f_i(x_j, E, u'_1, u'_2)$ for any $x_j \in [u''_j(c''_i), u''_j(c''_j)] = [u'_j(c_i), u'_j(c_j)]$ and any $i \in \{1, 2\}$, we have $u_i(\mu(c''_1, c''_2, E'', u''_1, u''_2)) \leq u_i(\mu(c_1, c_2, E, u'_1, u'_2))$ for $i \in \{1, 2\}$ by axiom IM. This implies that $\mu(c_1, c_2, E, u'_1, u'_2) = \mu(c''_1, c''_2, E'', u''_1, u''_2) = \gamma(c_1, c_2, E, u'_1, u'_2)$. That is, for the new claims problem $(c_1, c_2, E, u'_1, u'_2)$, the allocation solution μ and the fair allocation solution γ yield the same allocation outcome. By axiom INV, for the original claims problem (c_1, c_2, E, u_1, u_2), the two allocation solutions must also yield the same allocation outcome. □

Proof of Proposition 1:

We can find a positive affine transformation $T(x) = rx + s$ such that $(T \circ v_2)(c_1) = u_2(c_1)$ and $(T \circ v_2)(c_2) = u_2(c_2)$, i.e., $(T \circ k)(u_2(c_1)) = u_2(c_1)$ and $(T \circ k)(u_2(c_2)) = u_2(c_2)$.[18] Then we have $(T \circ v_2)(a^*) = (T \circ k)(u_2(a^*)) \geq u_2(a^*)$, where the equality follows from the fact that $a^* \in X$, and the inequality follows from the facts that $T \circ k$ is concave and strictly increasing, $u_2(c_1) \leq u_2(a^*) \leq u_2(c_2)$, $(T \circ k)(u_2(c_1)) = u_2(c_1)$ and $(T \circ k)(u_2(c_2)) = u_2(c_2)$.

Let $a^{*\prime\prime} = g^*(c_1, c_2, E, u_1, T \circ v_2)$. Since the claims problem (c_1, c_2, E, u_1, v_2) is deterministic, it must be true that $(c_1, c_2, E, u_1, T \circ v_2)$ is also deterministic and thus $a^{*\prime\prime} \in X$. Now, compare the claims problem (c_1, c_2, E, u_1, u_2) and $(c_1, c_2, E, u_1, T \circ v_2)$. Since $(T \circ v_2)(c_1) = u_2(c_1)$ and $(T \circ v_2)(c_2) = u_2(c_2)$, the line that connects the component-wise minimum and component-wise maximum of $(u_1(c_1), u_2(c_1))$ and $(u_1(c_2), u_2(c_2))$ is the same as the line that connects the component-wise minimum and component-wise maximum of $(u_1(c_1), (T \circ v_2)(c_1))$ and $(u_1(c_2), (T \circ v_2)(c_2))$. Now, notice that $(u_1(a^*), u_2(a^*))$ is on the

[18]We use $f \circ g$ to denote the composite function of f and g.

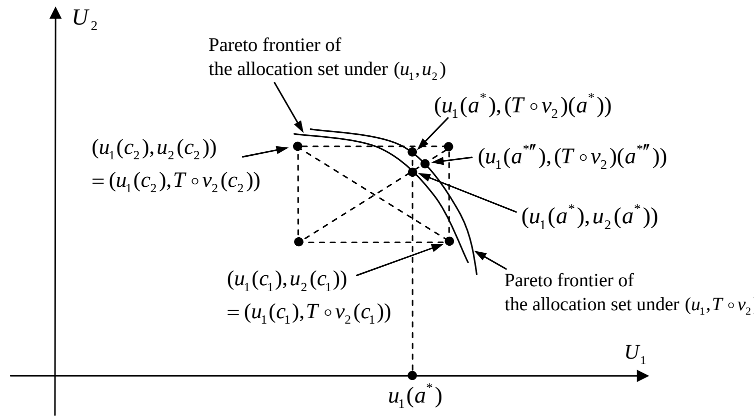


Figure 8: Comparison of two claims problems.

Pareto frontier of the allocation set $\{(u_1(x), u_2(x)) : x \in X\}$, then we must have that $(u_1(a^*), (T \circ v_2)(a^*))$ is on the Pareto frontier of the allocation set $\{(u_1(x), (T \circ v_2)(x)) : x \in X\}$. Refer to Figure 8. It is clear that $(T \circ v_2)(a^{*\prime\prime}) \leq (T \circ v_2)(a^*)$. That is, $(T \circ k)(u_2(a^{*\prime\prime})) \leq (T \circ k)(u_2(a^*))$. That is, $u_2(a^{*\prime\prime}) \leq u_2(a^*)$ because $T \circ k$ is strictly increasing. So, we have $u_2(a^{*\prime}) \leq u_2(a^*)$ because $a^{*\prime\prime} = a^{*\prime}$ by the invariance axiom. □

Proof of Theorem 2:

Let $C = (c_1, c_2, E, u_1, u_2) \in \mathcal{C}$ be a given claims problem. Let $(x_1, y_1) = (u_1(c_1), u_2(c_1))$ and $(x_2, y_2) = (u_1(c_2), u_2(c_2))$. Notice that we use x to represent the payoff obtained by player 1, and y to represent the payoff obtained by player 2.

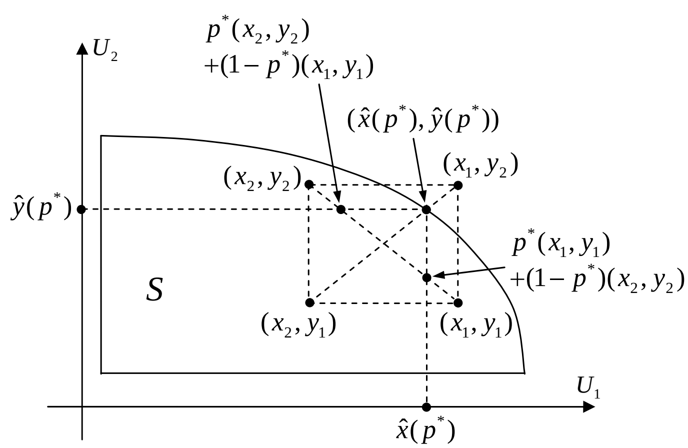


Figure 9: Determination of p^*.

Denote $\hat{x}(p) = px_1 + (1-p)x_2$ and $\hat{y}(p) = py_2 + (1-p)y_1$. Obviously, $\hat{x}(p)$ is player 1's payoff when player 1 requests that the arbitrator end the game, and $\hat{y}(p)$ is player 2's payoff when player 2 requests that the arbitrator end the game. Define p^* as the unique $p \in [0,1]$ such that $(\hat{x}(p), \hat{y}(p))$ lies on the Pareto frontier of bargaining set S (see Figure 9).[19] Let a^* be the lottery in $\Delta(X)$ such that $(u_1(a^*), u_2(a^*)) = (\hat{x}(p^*), \hat{y}(p^*))$. Assume that player 1 is chosen by the arbitrator at stage 0 (the case where player 2 is chosen by the arbitrator is similar). In the remainder of the proof, we show that it is a subgame-perfect equilibrium for player 1 to choose $p = p^*$ and offer a^*, which player 2 accepts.

Since stage 3 will never be reached in equilibrium, we start our analysis from stage 2.

(i) *Stage 2*

Let $m(p)$ be the lottery in $\Delta(X)$ such that $(u_1(m(p)), u_2(m(p))) = (\hat{x}(p), f(\hat{x}(p)))$. If the game moves to stage 2, then it is an equilibrium for player 2 to make the offer $m(p)$, which player 1 accepts. Player 2's equilibrium payoff is thus $f(\hat{x}(p))$.

(ii) *Stage 1*

We have the following two cases:

(1) The probability p is such that $(\hat{x}(p), \hat{y}(p)) \notin S$.

Let $c_1' \in \Delta(X)$ be the offer made by player 1 at the beginning of stage 1. Player 2 can either accept the offer or reject it. If player 2 rejects the offer, then player 2 can either request the arbitrator to end the game, obtaining a payoff of $\hat{y}(p)$, or move the game to stage 2, obtaining an equilibrium payoff of $f(\hat{x}(p))$. Since $(\hat{x}(p), \hat{y}(p)) \notin S$, we must have $\hat{y}(p) > f(\hat{x}(p))$ (see the left-hand figure in Figure 10). Thus, player 2's optimal action after rejecting player 1's offer c_1' is to request the arbitrator to end the game, and player 2 obtains a payoff of $\hat{y}(p)$. On the other hand, if player 2 accepts the offer c_1', then his payoff will be $u_2(c_1')$. As a result, player 1's offer c_1' will be accepted by player 2 if and only if $u_2(c_1') \geq \hat{y}(p)$. Let $n(p)$ be the lottery in $\Delta(X)$ such that $(u_1(n(p)), u_2(n(p))) = (f^{-1}(\hat{y}(p)), \hat{y}(p))$. It can be easily verified that player 1's optimal offer is $n(p)$, which will be accepted by player 2

[19] The probability p^* exists because $(x_1, y_2) \notin S$ (we have $(x_1, y_2) \notin S$ because c_1 and c_2 are incompatible).

immediately, and player 1 obtains a payoff of $f^{-1}(\hat{y}(p))$.[20]

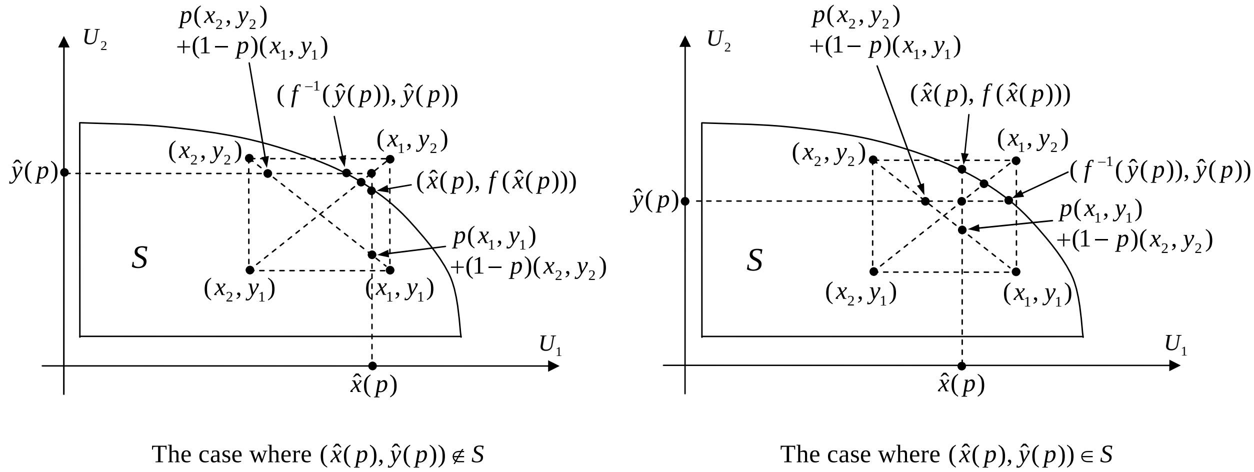


Figure 10: The choice of the probability p.

(2) The probability p is such that $(\hat{x}(p), \hat{y}(p)) \in S$.

Again, let $c_1' \in \Delta(X)$ be the offer made by player 1 at the beginning of stage 1. If player 2 rejects the offer, then player 2 can either request the arbitrator to end the game, obtaining a payoff of $\hat{y}(p)$, or move the game to stage 2, obtaining a payoff of $f(\hat{x}(p))$. Since $(\hat{x}(p), \hat{y}(p)) \in S$, we must have $\hat{y}(p) \leq f(\hat{x}(p))$ (see the right-hand figure in Figure 10). Thus, player 2's optimal action after rejecting player 1's offer c_1' is to move the game to stage 2, and player 2 obtains a payoff of $f(\hat{x}(p))$. On the other hand, if player 2 accepts the offer c_1', then player 2's payoff will be $u_2(c_1')$. As a result, player 1's offer c_1' will be accepted by player 2 if and only if $u_2(c_1') \geq f(\hat{x}(p))$. It can be easily verified that player 1's optimal offer is $m(p)$,[21] which will be accepted by player 2 immediately, and player 1 obtains a payoff of $\hat{x}(p)$.[22]

[20] If player 1 makes an offer c_1' with $u_2(c_1') < \hat{y}(p)$, then the offer will be rejected by player 2 and player 2 will request the arbitrator to end the game. Player 1's payoff is thus $px_2 + (1-p)x_1$, which is strictly less than $f^{-1}(\hat{y}(p))$ (see the left-hand figure in Figure 10).

[21] Recall that $m(p)$ is the lottery in $\Delta(X)$ such that $(u_1(m(p)), u_2(m(p))) = (\hat{x}(p), f(\hat{x}(p)))$.

[22] If player 1 makes an offer c_1' with $u_2(c_1') < f(\hat{x}(p))$, then the offer will be rejected by player 2 and player 2 will move the game to stage 2 in equilibrium. At stage 2, player 2 will make the offer $m(p)$, which player 1 accepts. Player 1's equilibrium payoff is thus $\hat{x}(p)$. As a result, player 1 is indifferent between offering $m(p)$, which will be accepted by player 2, and offering c_1' with $u_2(c_1') < f(\hat{x}(p))$, which will be rejected by player 2. For simplicity, we assume that player 1 makes the offer $m(p)$ in equilibrium. This assumption is made for simplicity and the relaxation of the assumption will not change the equilibrium payoff of the game.

(iii) *Stage 0*

Based on the analysis in (ii), player 1's payoff is maximized when $p = p^*$. Thus, it is subgame-perfect equilibrium for player 1 to choose $p = p^*$ at stage 0, and offer a^* at stage 1, and for player 2 to accept a^* immediately (recall that a^* is the lottery such that $(u_1(a^*), u_2(a^*)) = (\hat{x}(p^*), \hat{y}(p^*))$). Observing that for any $p \in [0, 1]$, $(\hat{x}(p), \hat{y}(p))$ lies on the line that connects (x_2, y_1) and (x_1, y_2), the point $(\hat{x}(p^*), \hat{y}(p^*))$ must be the intersection point of the Pareto frontier with the line that connects (x_2, y_1) and (x_1, y_2), i.e., $(\hat{x}(p^*), \hat{y}(p^*))$ must be the fair allocation solution payoff to the claims problem (c_1, c_2, E, u_1, u_2). In other words, the unique SPE payoff of the fair allocation procedure coincides with the fair allocation solution payoff. □

Proof of Theorem 3:

Recall that b_i is player i's maximal possible utility level from the allocation set S. Let d_i be player i's utility obtained from the disagreement point d, i.e., $d_1 = u_1(d)$ and $d_2 = u_2(d)$.

Stage 2

In the proof of Theorem 2, we analyzed the case where player 1's claim c_1 and player 2's claim c_2 are *incompatible*. Since in the strategic claim with fair allocation procedure, we allow players to make claims strategically, it is possible that the claims submit by the two players are *compatible*. The remainder of the proof mainly analyze the equilibrium behavior of players for the case where c_1 and c_2 are compatible, i.e., $(x_1, y_2) \in S$ (recall that $(x_1, y_1) = (u_1(c_1), u_2(c_1))$ and $(x_2, y_2) = (u_1(c_2), u_2(c_2))$). We have the following two subcases.[23]

(i) $x_1 \geq x_2$.

In this case, we have two possibilities (see Figure 11): $y_1 \leq y_2$ and $y_1 > y_2$. In both cases, it can be readily verified that in equilibrium, player 1 chooses $p = 1$ and proposes the offer a, where a is such that $(u_1(a), u_2(a)) = (x_1, f(x_1))$. Player 2 accepts a immediately.

[23]Given that the analysis of the case of compatible claims is similar to the case of incompatible claims, we will only provide a sketch of the proof below.

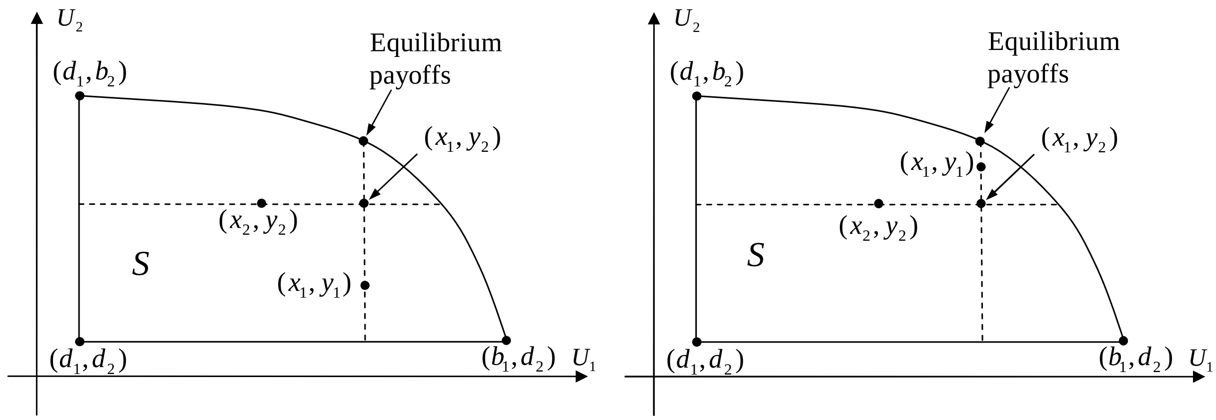


Figure 11: The case where $x_1 \geq x_2$.

(ii) $x_1 < x_2$.

In this case, we have three possibilities (see Figure 12): $y_1 \leq y_2$, $y_2 < y_1 \leq f(x_2)$ and $y_1 > f(x_2)$. As regards the first two possibilities, it can be readily verified that in equilibrium, player 1 chooses $p = 0$ and proposes the offer $a \in \Delta(X)$, where a is such that $(u_1(a), u_2(a)) = (x_2, f(x_2))$. Player 2 accepts a immediately. As regards the last possibility, it can be verified that in equilibrium player 1 chooses p^*, where p^* is the unique $p \in [0, 1]$ that satisfies $(\hat{x}(p), \hat{y}(p)) \in P(S)$,[24] and player 1 proposes the offer a, where a is such that $(u_1(a), u_2(a)) = (\hat{x}(p^*), \hat{y}(p^*))$. Player 2 accepts a immediately.

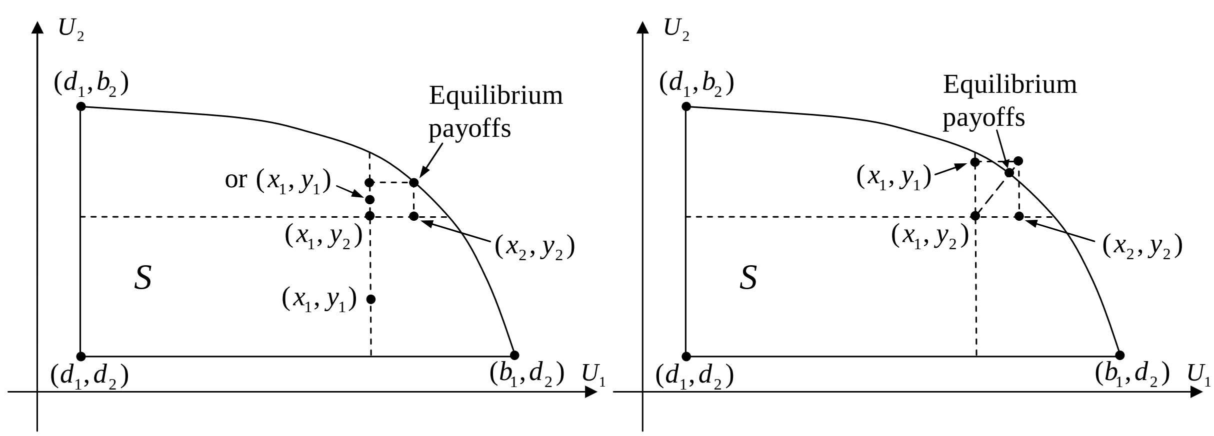


Figure 12: The case where $x_1 < x_2$.

Stage 1

[24] Recall that $\hat{x}(p) = px_1 + (1 - p)x_2$ and $\hat{y}(p) = py_2 + (1 - p)y_1$.

We now return to stage 1. In all the cases mentioned above (including the case where the players' claims are incompatible), player 1 obtains a greater stage-2 equilibrium payoff as his stage-1 claim (measured in terms of utility level) $(x_1, y_1) \in S$ moves from the upper-left to the lower-right. Player 2 obtains a greater stage-2 equilibrium payoff as his (stage-1) claim (measured in terms of utility level) $(x_2, y_2) \in S$ moves from the lower-right to the upper-left. As a result, in equilibrium, player 1 must claim (measured in terms of utility level) (b_1, d_2) and player 2 must claim (measured in terms of utility level) (d_1, b_2) at stage 1, and the equilibrium outcome of the entire game coincides with the Kalai-Smorodinsky solution outcome. □

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