Fair Allocation When Players' Preferences are Unknown

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Abstract: Suppose an arbitrator needs to allocate an asset among two players, whose claims on the asset are incompatible. The allocation outcome is said to be *fair* if the arbitrator awards an outcome that brings the same utility payoff to the two players whenever the two players' claims are symmetric and the allocation set is symmetric. In conjunction with other natural axioms, this fairness requirement implies a unique allocation outcome for any claims problem. We propose a mechanism which can be used by the arbitrator to implement this allocation outcome, even when the players' preferences are *unknown* to the arbitrator.

Keywords: Fair allocation; Claims problem; Implementation; Axioms; Kalai-Smorodinsky solution.

JEL classification: C78, D63, J52

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Two hold a garment; one claims it all, the other claims half. What is an equitable division of the garment? (Babylonian Talmud, Baba Metzia 2a)

1 Introduction

Suppose there are two players, each of whom claims part of or all of an asset. The claims of both players are legal and verifiable. However, the two players' claims are incompatible with each other, in the sense that the two players' demands cannot be satisfied simultaneously. The question is, how should the asset be divided among the two players, in the event that an arbitrator is called in? Examples of such situations include (i) how should a firm's liquidation value be divided among its creditors when the firm goes bankrupt? and (ii) how should the estate of a person be divided among his heirs, if the heirs to the estate are willed more than the value of the estate?

In the literature, many solutions have been proposed to deal with the *claims problem* above. The most well-know solution is probably the *proportional solution*. It requires that a player obtains an amount that is proportional to the player's claim. In the classic example in the Talmud where player 1 claims all of a good and player 2 claims half of the good, the proportional solution awards 2/3 to player 1 and 1/3 to player 2. Another well-known solution is the *contested garment solution*. It requires that a player be awarded an amount equal to the player's uncontested portion of the asset (i.e., the excess of the total asset over the player's opponent's claim) plus half of the contested portion of the asset (i.e., the excess of the total asset over the sum of the two players' uncontested portions). In the Talmud example, the contested garment solution awards 3/4 to player 1 and 1/4 to player 2.

A common feature of the proportional solution, the contested garment solution and many other solutions in the literature is that players are awarded different amounts because their claims differ (in other words, the two players are awarded the same amount provided that the two players' claims are the same). This is natural, because players' claims are legal and verifiable, and the arbitrator should respect the players' claims. However, what is missing in the literature is that players usually also differ in their utilities over the allocation of the asset, and the various solutions in the literature usually do not take this into consideration. Let's consider a simple example where both players claim all of a (perfectly divisible) good, and the two players' utilities are such that both players attach the same value for the first half of the good, but for the second half of the good, player 1 attaches a much higher value than player 2 (more precisely, player 2 is more risk averse than player 1). Both the proportional solution and the contested garment solution suggest (1/2,1/2) to be the outcome, regardless of players' utilities. However, obtaining half of the good means a lot of suffering for player 1 (considering his claim is all of the good), while it means little suffering for player 2. A more *reasonable* solution is probably to award player 1 more than half, and player 2 less than half. This kind of intuition suggests that players' utilities may play a role in the determination of the fair allocation in a claims problem.

More specifically, we will introduce players' utilities into claims problems by imposing the following requirement on the solution: an arbitrator awards an outcome that brings the two players the same payoff whenever the two players' claims are symmetric and the allocation set (i.e., the set of all possible divisions of the asset between the two players) is symmetric, where both the claims and the allocation set are measured in terms of players' utilities. This requirement appears to be a minimal and most uncontroversial requirement in order for a solution to be fair. We thus call this requirement the *fairness* requirement. This requirement is a very weak restriction because it provides no prediction for the cases where either the players' claims are not symmetric or the allocation set is not symmetric. We thus need to impose some other requirements on the solution. Actually, we impose three more requirements and they are Scale Invariance, Pareto Optimality, and Individual Monotonicity. Notice that all the four requirements above are borrowed from the bargaining literature (see, e.g., Nash (1950), Kalai and Smorodinsky (1975)), but we give them (especially, the fairness requirement) new interpretations in the context of claims problems.

This paper attempts to answer the following two questions. First, does there exist a solution[1] that satisfies the fairness requirement and the other three requirements? Second, and more importantly, if there exists a solution that satisfies the four requirements, but the arbitrator is ignorant of players' utilities, can we design a mechanism such that the arbitrator can use it to implement the solution we find?

The main results of the paper are as follows. First, we find that there is a unique solution that satisfies all the four requirements. We call this solution the *fair allocation solution*. Second, we find a mechanism, called the *fair allocation procedure*, which implements the fair allocation solution in subgame-perfect equilibrium. That is, for any claims problem, the unique subgame-perfect equilibrium outcome of the fair allocation procedure coincides with the fair allocation solution outcome. Third, as a byproduct of our research, we find a mechanism that implements the Kalai-Smorodinsky bargaining solution (the mechanism is a variant of the fair allocation procedure).

We next briefly explain how the fair allocation solution is determined. Assume that a player's claim for himself cannot exceed the total amount of the asset available. Let c_i be the allocation in which player i's claim is just satisfied, c_j be the allocation in which player j's claim is just satisfied, and a be the final allocation imposed by the arbitrator. We call $\frac{u_i(c_i) - u_i(a)}{u_i(c_i) - u_i(c_j)}$ player i's *normalized utility loss*. A player's normalized utility loss measures the player's utility loss from the player's claim to the final allocation (i.e., $u_i(c_i) - u_i(a)$), scaled by the inverse of the player's maximum possible utility loss ($u_i(c_i) - u_i(c_j)$). We show that the fair allocation solution is the unique allocation on the Pareto frontier of the allocation set such that the two players' normalized utility losses are identical. Obviously, the players' risk attitudes play an important role in the determination of the fair allocation solution. We find that as a player becomes more risk averse, the player obtains less fair allocation solution payoff.[2]

The fair allocation procedure we find is a variant of the two-stage alternating-offer game,

[1] Formally, a *solution* is a function that assigns an outcome in the allocation set to each claims problem.

[2] In addition, when both players are risk neutral, the fair allocation solution coincides with the equal (physical) losses solution, which also coincides with the contested garment solution.

and is as follows. At the beginning of the game, the arbitrator chooses a player at random, and asks the chosen player to choose a probability $p \in [0,1]$. The two players then make offers to their opponents sequentially, with the player who was chosen at the beginning of the game being the first proposing player. At any stage, the responding player can either accept the proposing player's offer or reject it. In addition, the arbitrator can end the game whenever a responding player has just rejected a proposing player's offer and the responding player requests to do so. We assume that if the arbitrator is requested by a responding player, say player i, to end the game, then the game ends with the lottery where player i's claim is implemented with probability p and player j's claim is implemented with probability $1-p$ (i.e., $p \cdot c_i + (1-p) \cdot c_j$ is the outcome).

It can be shown that the unique subgame-perfect equilibrium outcome of the fair allocation procedure coincides with the fair allocation solution outcome for any claims problem. Moreover, the fair allocation procedure is independent of players' utilities and the arbitrator does not require any knowledge of players' utilities in order to enforce this procedure. Our research thus provides a *non-cooperative* foundation for the fair allocation solution.

Related literature. Our paper is mostly related to the literature about *claims problems* (see Thomson (2003) for a survey of the literature). The claims problem defined in this paper is more general than the (standard) claims problem in the literature because our claims problem takes players' utilities into consideration. However, there are some exceptions in the literature in which players' utilities are also considered. In particular, Chun and Thomson (1992) studied the so-called *bargaining problems with claims*, in which the two players' claims serve as the "reference points" for the two players to bargain with each other (and if the two players cannot reach agreement, then the disagreement point will be enforced). Mariotti and Villar (2005) studied the *Nash rationing problems*, which can be regarded as the translation of the Nash bargaining problem to a rationing (or claims problem) scenario, and they found that there is a multivalued solution that is characterized by an extension of Nash's axioms.

Our paper is also related to Rong (2012). Actually, the axiomatic part of this paper mostly builds on Rong (2012), and the fair allocation solution coincides with the *symmetric arbitration solution* proposed in Rong (2012). However, the contexts behind the two solutions are different. The game considered in Rong (2012) is a variant of the Nash demand game (Nash (1953)), while the problem considered in this paper is a claims problem. Also, the equal normalized utility loss interpretation of the fair allocation solution is very different from the geometric interpretation of the symmetric arbitration solution in Rong (2012). Lastly, and also most importantly, Rong (2012) mainly focuses on the *characterization* of the symmetric arbitration solution, while this paper mainly focuses on the *implementation* of the fair allocation solution.

This paper is organized as follows. Section 2 presents the basic elements of the model. Section 3 studies the role of players' risk attitudes in the determination of the fair allocation solution. Section 4 discusses the mechanism that implements the fair allocation solution. Section 5 discusses the mechanism that implements the Kalai-Smorodinsky solution. Concluding remarks are offered in Section 6.

2 Preliminaries

2.1 The model

Suppose there is a perfectly divisible asset to be divided among two players. Let $X = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y \leq E\}$ be the set of all possible *deterministic* allocations of the asset among the two players, where x represents player 1's allocation, y represents player 2's allocation and E is the total amount of the asset available. Let $P(X)$ be the Pareto frontier of X, i.e., $P(X) = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y = E\}$. We allow randomization in the allocation. That is, an allocation can be a lottery over deterministic allocations in X. We use $\Delta(X)$ to denote the set of all lotteries over X, and call $\Delta(X)$ the *allocation set*. Let $u_i : \Delta(X) \rightarrow R$ denote player i's *expected utility* function. We assume that u_i only depends on player i's own allocation, and when restricted in X, u_i is continuous and strictly increasing in player i's own allocation. Let $S(\Delta(X), u_1, u_2) = \{(u_1(a), u_2(a)) : a \in \Delta(X)\}$ be the image of $\Delta(X)$ under the utility functions (u_1, u_2). $S(\Delta(X), u_1, u_2)$ is thus the allocation set measured in terms of players' utilities.[3] For simplicity, we write $S(\Delta(X), u_1, u_2)$ as S whenever there is no confusion. The Pareto frontier of S is denoted by $P(S) = \{s \in S | s' >> s \Rightarrow s' \notin S\}$. Notice that S is a convex set because $\Delta(X)$ is a set of lotteries and u_1 and u_2 represent expected utilities. We use d to denote the allocation $(0, 0)$. A typical allocation set is illustrated in Figure 1.

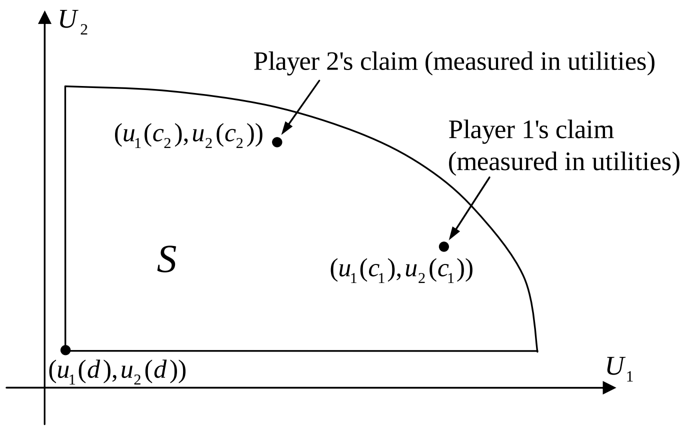

Figure 1: Allocation set and incompatible claims.

We use $c_1 = (c_{11}, c_{12})$ to denote player 1's *claim*, $c_2 = (c_{21}, c_{22})$ to denote player 2's *claim*, and assume that $c_1 \in P(X)$ and $c_2 \in P(X)$. In our definition, player i's claim is an allocation plan, which not only includes the player's demand (c_{ii}) for himself, but also includes the player's suggested allocation (c_{ij}) for his opponent. Although we define a player's claim as a two-dimensional vector, rather than a one-dimensional value, which is usually the case in the literature, the meaning of our definition is essentially the same as the definition in the literature because we require a player's claim be on the Pareto frontier of X. That is, for player i, as long as the player's demand for himself is given, then his suggested allocation for his opponent is automatically determined.[4] Notice that although we require

[3] In the remainder of the paper, the "allocation set" either refers to the set $\Delta(X)$, or the set $S(\Delta(X), u_1, u_2)$. The exact meaning is usually clear in the context in which it appears.

[4] The main reason that we use a player's allocation plan, rather than the player's own demand, to describe a player's claim is that it will be easier to state the fair allocation solution, which we will define shortly in the next subsection.

the two players' (physical) claims be on the Pareto frontier of X, the two players' utility claims may not lie on the Pareto frontier of S (see Figure 1).[5]

We assume that the two players' claims are *legal* and *verifiable*, so that a player cannot strategically report his claim. In addition, we assume that the two players' claims are *incompatible*, meaning that there is no allocation $a \in \Delta(X)$ such that $(u_1(c_1), u_2(c_2)) = (u_1(a), u_2(a))$.

We call (c_1, c_2, E, u_1, u_2) a *claims problem*, and use $\mathcal{C}$ to denote the set of all claims problems that satisfy the assumptions mentioned above. A *solution* is a mapping $g : \mathcal{C} \rightarrow \Delta(X)$ that associates a lottery in $\Delta(X)$ with each claims problem. A *mechanism* (or, *game form*) Γ is a triple $(\Sigma_1, \Sigma_2; h)$, where Σ_i is the strategy set of player i and Σ_i is independent of the players' utilities, and $h : \Sigma_1 \times \Sigma_2 \rightarrow \Delta(X)$ is an outcome function (Dagan and Serrano (1998); Trockel (2002)). A mechanism Γ is said to *implement* a solution g in subgame-perfect equilibrium if for any claims problem $C \in \mathcal{C}$, $g(C)$ is the unique pure strategy subgame-perfect equilibrium outcome of the game (Γ, C).

2.2 Fairness requirement on the solution

We say a solution is *fair* if it assigns the same payoff to the two players whenever the two players' claims (measured in terms of utility level) are symmetric and the allocation set (also measured in terms of utility level) is symmetric. That is, a solution g is fair if it satisfies the following axiom.

- **Fairness** (F): For any claims problem $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$ where $u_1(c_1) = u_2(c_2)$, $u_2(c_1) = u_1(c_2)$ and S is symmetric, we have $u_1(g(c_1, c_2, E, u_1, u_2)) = u_2(g(c_1, c_2, E, u_1, u_2))$.

[5]This occurs, for example, when both players are risk seeking (i.e., for $i = 1, 2$, u_i is a convex function in player i's own allocation when the function is restricted in the domain X). In this case, the image of $P(X)$ under the two players' utilities is a convex curve in R^2, while the Pareto frontier of S is a line which lies above the image of $P(X)$ under the two players' utilities.

It is obvious that the fairness requirement is insufficient to determine any solution. We thus impose three additional natural axioms on the solution. They are Scale Invariance, Pareto Optimality, and Individual Monotonicity. Scale Invariance requires that the (physical) solution outcome be the same under positive affine transformations of players' utilities. The idea behind this axiom is that the allocation outcome should only depend on players' preferences and not on the precise utility representations of players. Since under the expected utility assumption, a player's utility is unique up to positive affine transformations, the (physical) allocation outcome should be invariant to any positive affine transformation of players' utilities. Pareto optimality requires that the solution outcome be on the Pareto frontier of the allocation set for any claims problem. Individual Monotonicity requires that a player obtain a greater allocation payoff when the part of the Pareto frontier lying between the two players' claims expands from the viewpoint of the player, with the two players' claims being fixed.[6]

Let g be a solution. The axioms Scale Invariance, Pareto Optimality, and Individual Monotonicity are defined as follows.

- **Scale Invariance** (INV): If $T : R^2 \to R^2$ represents a positive affine transformation, i.e., $T(x, y) = (r_1 x + s_1, r_2 y + s_2)$ for some positive constant r_i and some constant s_i, then we have $g(c_1, c_2, E, T(u_1, u_2)) = g(c_1, c_2, E, u_1, u_2)$ for any $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$.

- **Pareto Optimality** (PO): For any claims problem $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$, we have $(u_1(g(c_1, c_2, E, u_1, u_2)), u_2(g(c_1, c_2, E, u_1, u_2))) \in P(S)$.

- **Individual Monotonicity** (IM): For any two claims problems $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$ and $(c'_1, c'_2, E', u'_1, u'_2) \in \mathcal{C}$, if $u_1(c_1) = u_1(c'_1)$, $u_2(c_1) = u_2(c'_1)$, $u_1(c_2) = u_1(c'_2)$, $u_2(c_2) = u_2(c'_2)$, and $f_i(x_j, E, u_1, u_2) \leq f_i(x_j, E', u'_1, u'_2)$ for any $x_j \in [u_j(c_i), u_j(c_j)]$,

[6]Notice that Individual Monotonicity is a weaker requirement than Strong Monotonicity defined in Rong (2012), which requires that a player obtain a greater allocation payoff as the entire allocation set expands (Individual Monotonicity only requires that "part of" the allocation set expands while the other part of the allocation set might actually shrink).

then $u_i(g(c_1, c_2, E, u_1, u_2)) \leq u_i(g(c'_1, c'_2, E', u'_1, u'_2))$ (where $f_i(x_j, E, u_1, u_2)$ is the maximal possible payoff for i in S given that player j's payoff is x_j).

We obtain the following result.

Theorem 1. *There is a unique solution,*[7] *called the* ***fair allocation solution****, which satisfies F, INV, PO, and IM. Let $a^* \in \Delta(X)$ be the fair allocation solution outcome to a given claims problem $(c_1, c_2, E, u_1, u_2) \in \mathcal{C}$, then $(u_1(a^*), u_2(a^*))$ must be the unique point on the Pareto frontier of the allocation set S such that $\dfrac{u_1(c_1) - u_1(a^*)}{u_1(c_1) - u_1(c_2)} = \dfrac{u_2(c_2) - u_2(a^*)}{u_2(c_2) - u_2(c_1)}$.*

The ratio $\dfrac{u_i(c_i) - u_i(a^*)}{u_i(c_i) - u_i(c_j)}$ represents player i's *normalized utility loss*, since $u_i(c_i) - u_i(a^*)$ is player i's utility loss from his claim point c_i to the allocation outcome a^* and $u_i(c_i) - u_i(c_j)$ is the maximum possible utility loss that player i can have (because the worst case for player i is to have player j's claim as the allocation outcome). The fair allocation solution thus simply requires that the two players' normalized utility losses be identical. We next briefly illustrate the calculation of the fair allocation solution using a simple example (see also Example 1 in the end of this section). Suppose that the set of deterministic allocations $X = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y \leq 1\}$, and that player 1's claim is $c_1 = (1, 0)$ and player 2's claim is $c_2 = (\frac{1}{2}, \frac{1}{2})$. If both players are risk neutral (in particular, assuming that $u_1(x, y) = x$ and $u_2(x, y) = y$), then the fair allocation solution outcome is $a^* = (\frac{3}{4}, \frac{1}{4})$, because $\dfrac{u_1(c_1) - u_1(a^*)}{u_1(c_1) - u_1(c_2)} = \dfrac{u_2(c_2) - u_2(a^*)}{u_2(c_2) - u_2(c_1)} = \dfrac{1}{2}$ (note that $u_1(c_1) - u_1(c_2) = u_2(c_2) - u_2(c_1) = \frac{1}{2}$ and $u_1(c_1) - u_1(a^*) = u_2(c_2) - u_2(a^*) = \frac{1}{4}$).

The fair allocation solution is illustrated in Figure 2. The figure shows that the fair allocation solution is also the intersection point of the Pareto frontier of the allocation set and the line joining the component-wise minimum and the component-wise maximum of the two players' claims.

[7] The solution in unique up to payoff. That is, any two fair allocation solutions must yield the same payoffs for the two players for any given claims problem.

The proof of Theorem 1 is omitted because the idea of the proof is similar to Kalai and Smorodinsky (1975) and Rong (2012). The intuition of the proof, however, is as follows. For any asymmetric claims problem, we can always find a positive affine transformation to transform it into a new claims problem, in which the two players' claims in the new problem are symmetric. The allocation set of this new claims problem may still be asymmetric. However, we can always find a symmetric allocation set, which is contained in the above asymmetric allocation set, and determine the solution for this smaller claims problem using the fairness requirement and Pareto Optimality. Finally, using Individual Monotonicity and Scale Invariance allows us to determine the solution to the original problem.

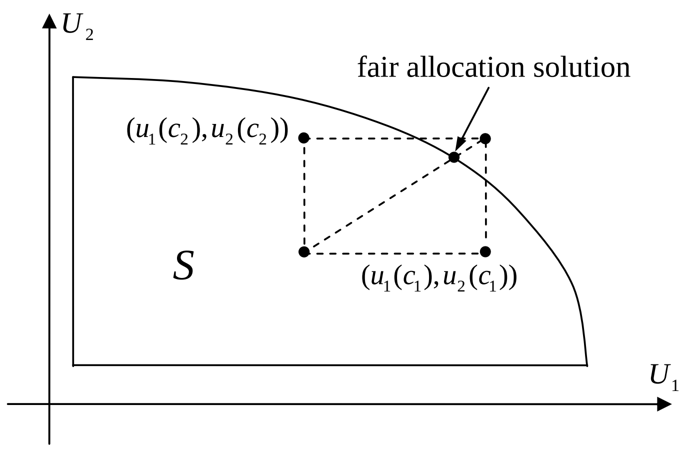

Figure 2: The fair allocation solution.

The following example illustrates how the fair allocation solution outcome is determined for the simple case where there is a unit of asset with player 1 claiming all of the asset and player 2 claiming half of the asset. We allow players' utilities to be either risk-neutral or risk-averse.

Example 1. *Suppose the set of deterministic allocations* $X = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y \leq 1\}$. *Suppose player 1's claim is* $(1, 0)$, *and player 2's claim is* $(\frac{1}{2}, \frac{1}{2})$. *Obviously, the two players' claims are incompatible.*

If the two players' utilities on X *are such that* $u_1(x, y) = x$ *and* $u_2(x, y) = y$ *for any* $(x, y) \in X$ *(i.e., both players are risk neutral), then it can be shown that the fair allocation* *solution outcome is* $(\frac{3}{4}, \frac{1}{4})$. *The normalized utility losses of both players are 1/2.*

If the two players' utilities on X *are such that* $u_1(x, y) = x$ *and* $u_2(x, y) = \sqrt{y}$ *for any* $(x, y) \in X$, *then the fair allocation solution outcome is* $(0.809, 0.191)$. *The normalized utility losses of both players are 0.382.*

If the two players' utilities on X *are such that* $u_1(x, y) = \sqrt{x}$ *and* $u_2(x, y) = y$ *for any* $(x, y) \in X$, *then the fair allocation solution outcome is* $(0.739, 0.261)$. *The normalized utility losses of both players are 0.478.*

Example 1 shows that as a player's preference changes from risk neutral preference to risk averse preference, the player's fair allocation solution outcome becomes worse for the player. As will be illustrated in the next section, this result turns out to be true for a very general class of claims problems.

3 Risk sensitivity in fair allocation

This section shows that as long as the claims problems are deterministic, then risk aversion is disadvantageous for a player under the fair allocation solution. We say that a claims problem (c_1, c_2, E, u_1, u_2) is *deterministic* if the fair allocation solution outcome to the problem is a deterministic allocation. The set of deterministic claims problems are very general. It includes, for example, problems where (i) the Pareto frontier of the allocation set S is strictly concave, or (ii) there is no randomization, i.e., the allocation set is X, rather than $\Delta(X)$.

Let u_i and v_i be two utility functions defined on $\Delta(X)$. We call v_i is *more risk averse* than u_i if there exists a concave and strictly increasing function k such that $v_i(x) = k(u_i(x))$ for any $x \in X$.

We use g^* to denote the fair allocation solution. We obtain the following result.

Theorem 2. *Suppose there are two claims problems* (c_1, c_2, E, u_1, u_2) *and* (c_1, c_2, E, u_1, v_2), *where both problems are deterministic and* v_2 *is more risk averse than* u_2. *Then we have*

$u_2(a^*) \geq u_2(a^{*\prime})$, *where* $a^* = g^*(c_1, c_2, E, u_1, u_2)$, $a^{*\prime} = g^*(c_1, c_2, E, u_1, v_2)$.

Proof: see the appendix. □

In order to illustrate the intuition of the proof of Theorem 2, let us assume that the two players' (utility) claims are on the Pareto frontier (refer to Figure 3).[8] It is without loss of generality to assume that $v_2(c_1) = u_2(c_1)$ and $v_2(c_2) = u_2(c_2)$ (otherwise, we can apply a positive affine transformation to v_2 such that the two equalities hold). Given that v_2 is more risk averse than u_2, the Pareto frontier of the allocation set under (u_1, v_2) is more "bowed-out" than the Pareto frontier of the allocation set under (u_1, u_2). This implies that player 2 obtains a smaller payoff when player 2's utility is v_2 than when player 2's utility is u_2 (i.e., $v_2(a^{*\prime}) < v_2(a^*)$ or $u_2(a^{*\prime}) < u_2(a^*)$ where a^* is the fair allocation solution outcome when player 2's utility is u_2 and $a^{*\prime}$ is the fair allocation solution outcome when player 2's utility is v_2).

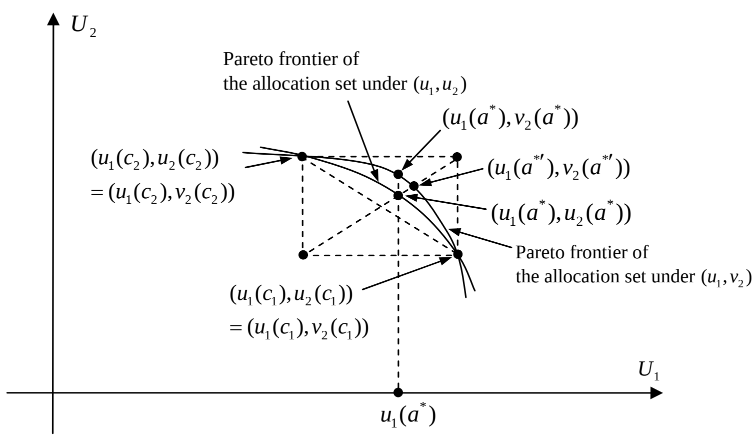

Figure 3: Comparison of two claims problems – an illustration.

The intuition of Theorem 2 can be explained as follows. The fair allocation solution requires that the two players' normalized utility losses be identical. If player 2 becomes more risk averse, then player 2's normalized utility loss, when evaluated at the original fair allocation outcome, becomes smaller (i.e., $\frac{v_2(c_2) - v_2(a^*)}{v_2(c_2) - v_2(c_1)} \leq \frac{u_2(c_2) - u_2(a^*)}{u_2(c_2) - u_2(c_1)}$). In order to

[8]This assumption is made for the purpose of exposition, and it is not required in the formal proof of Theorem 2.

re-balance the two players' normalized utility losses, player 2 thus needs to give up more. As a result, player 2 obtains less payoff when his utility becomes more risk averse.

Finally, notice that when players preferences are risk neutral, equal normalized utility losses implies equal physical losses. So, under risk-neutral preferences, our fair allocation solution coincides with the equal (physical) losses solution (which also coincides with the contested garment solution in the 2-person claims problem).

4 Fair allocation procedure

In this section, we assume that the arbitrator does not know the players' preferences.[9] We propose an arbitration procedure known as the *fair allocation procedure* to implement the fair allocation solution. The difficulty of finding a mechanism to implement a solution is that such a mechanism should not rely on players' preferences, so the mechanism can be used by the arbitrator even when the arbitrator is ignorant of players' preferences.

We use $p \cdot a_1 + (1-p) \cdot a_2$ to denote the lottery that assigns probability p to the allocation $a_1 \in \Delta(X)$ and assigns probability $1-p$ to the allocation $a_2 \in \Delta(X)$. The *fair allocation procedure* is defined as follows.

- Stage 0:
  - The arbitrator chooses a player at random. Suppose player i is chosen.
- Stage 1:
  - Player i chooses a probability $p \in [0, 1]$, and then makes an offer to player j. This offer is denoted by $c_i' \in \Delta(X)$.
  - Player j chooses whether to accept the offer or reject it. If player j chooses to accept the offer, then c_i' is the outcome. Otherwise, player j can choose one of

[9] We still assume that the players know each other's preferences. A typical example that fits this assumption is the King Solomon's dilemma, in which the two women knew each other's preference, but the King Solomon did not know the two women's preferences.

the following two options: (i) ask the arbitrator to end the game, and (ii) move the game to the next stage.

- Stage 2:
  - Player j makes an offer to player i. This offer is denoted by $c'_j \in \Delta(X)$.
  - Player i chooses whether to accept the offer or reject it. If player i chooses to accept the offer, then c'_j is the outcome. Otherwise, player i can choose one of the following two options: (i) ask the arbitrator to end the game, and (ii) move the game to the next stage.
- Stage 3:
  - $(0, 0)$ is the outcome.

We assume that if the arbitrator is asked by player i to end the game, then the arbitrator chooses the lottery $p \cdot c_i + (1-p) \cdot c_j$ as the arbitration outcome. Similarly, if the arbitrator is asked by player j to end the game, then the arbitrator chooses the lottery $p \cdot c_j + (1-p) \cdot c_i$ as the arbitration outcome. That is, the arbitrator always chooses an outcome that is a compromise between the two players' claims, where the weights on the two players' claims depend on (i) the probability p, and (ii) the person who makes the request to end the game.

The fair allocation procedure can be viewed as a variant of a two-stage alternating-offer game. A novel feature of the fair allocation procedure is that whenever a player rejects his opponent's offer, the player can ask the arbitrator to end the game. This additional option that is available to a responding player can be regarded as the player's *outside option*. We obtain the following result.

Theorem 3. *For any given claims problem $C \in \mathcal{C}$, the unique SPE payoff of the fair allocation procedure coincides with the fair allocation solution payoff.*

Proof: see the appendix. □

Assuming that player 1 is chosen at stage 0, the intuition of Theorem 3 can be explained as follows. The key to the equilibrium analysis of the fair allocation procedure is to determine the probability p that player 1 will choose at the beginning of stage 1. Obviously, if p is large, then both players have good outside options. If p is small, then both players have bad outside options. In equilibrium, p cannot be too large, because otherwise, player 1 will have to make a very favorable offer to player 2 since player 2 has a very good outside option. On the other hand, p cannot be too small, because otherwise, player 1 still must make a very favorable offer to player 2. This is because if player 1's offer is not sufficiently favorable to player 2, then player 2 will reject player 1's offer and make a very unfavorable offer to player 1, and player 1 will have to accept such an offer since player 1 does not have a good outside option. It turns out that in equilibrium, p must be such that player 2 is indifferent between his two options upon rejection: asking the arbitrator to end the game (i.e., ending the game with player 2's outside option) and moving the game to the next stage. With this delicate choice of p, player 1 will make an equilibrium offer that is exactly the same as the fair allocation solution outcome, and the offer will be accepted by player 2 immediately.

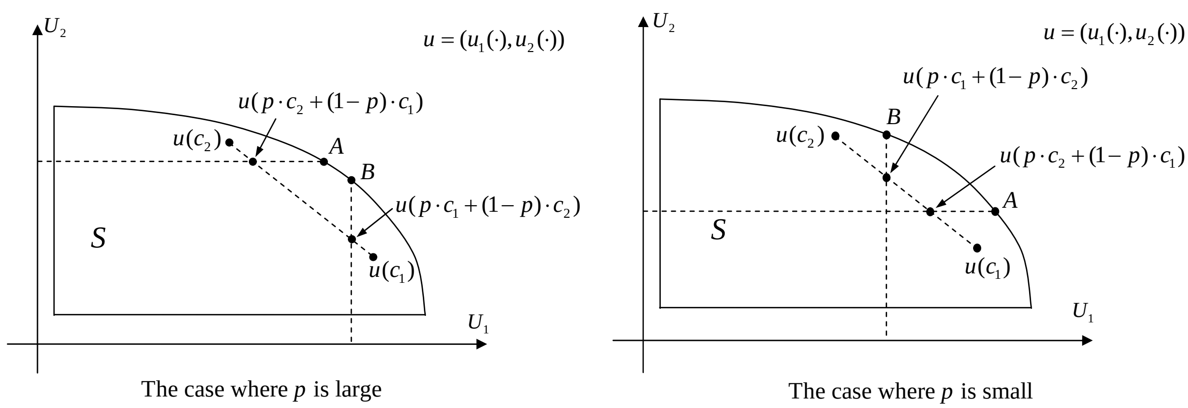

Figure 4: The cases where p is large or small.

Figure 4 and Figure 5 further illustrate the observations above. The left panel in Figure 4 illustrates the case where the probability p is large. In this case, each player's outside option is close to the player's claim, and the equilibrium of the game is that player 1 makes an offer at A, which player 2 accepts. The right panel in Figure 4 illustrates the case where the probability p is small. In this case, each player's outside option is distant from the player's claim, and the equilibrium is that player 1 makes an offer at B, which player 2 accepts. Obviously, player 1's payoff is maximized when the probability p is such that A and B coincide with each other (see Figure 5). In this case, the equilibrium offer (a^*) that player 1 makes is exactly the same as the fair allocation solution outcome.

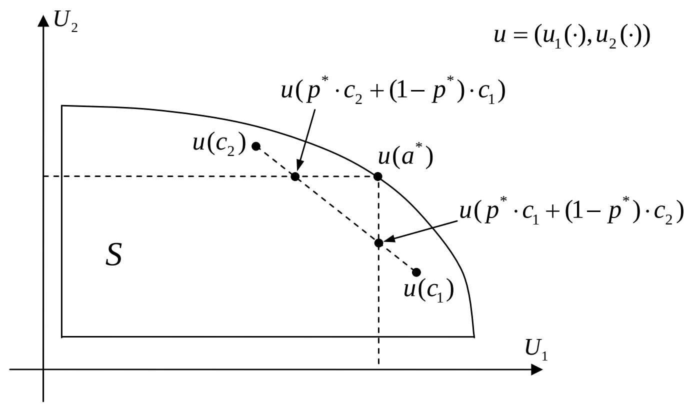

Figure 5: The equilibrium p and the equilibrium offer.

5 Implementation of the Kalai-Smorodinsky solution

Up to this point, we have assumed that both players' claims are legal and verifiable, and that players cannot "choose" their claims. In this section, we assume that players' claims are *not* verifiable and thus players can strategically report their claims. We can also understand this situation as a bargaining situation (e.g., a Nash demand game in which players can make arbitrary demands).

We call $(d, \Delta(X), u_1, u_2)$ a *bargaining problem*, where $d = (0,0)$ is the *disagreement point*, $\Delta(X)$ is the *bargaining set*, and u_1 and u_2 are the two players' expected utilities. We use b_i to denote player i's maximal possible utility level from the bargaining set S. The *Kalai-Smorodinsky solution* to the bargaining problem $(d, \Delta(X), u_1, u_2)$ is determined by the intersection point of the Pareto frontier of S and the line joining $(u_1(d), u_2(d))$ and (b_1, b_2). If we normalize $(u_1(d), u_2(d)) = (0, 0)$, then the Kalai-Smorodinsky solution simply requires that each player obtain a utility payoff that is proportional to the player's maximal possible utility payoff.

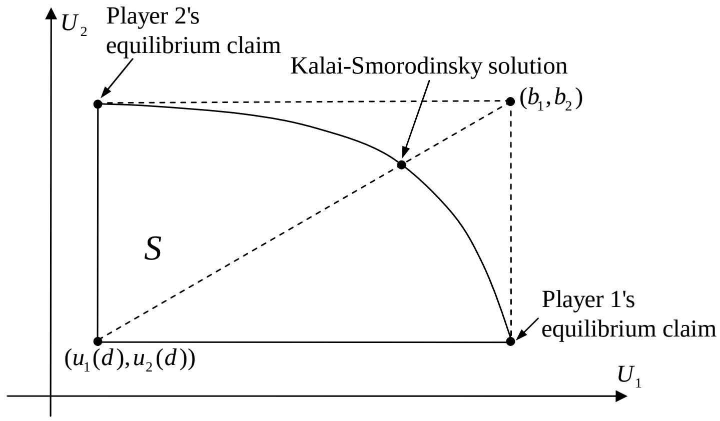

Figure 6: Implementation of the Kalai-Smorodinsky Solution.

Notice that a property of our fair allocation solution is that as a player claims more for himself, then the player's fair allocation solution payoff increases. As a result, if both players are allowed to strategically submit their claims to the arbitrator and both players know that the arbitrator will use the fair allocation solution to determine the outcome, then each player has an incentive to submit a claim that is as extreme as possible. Eventually, both players will submit extreme claims, and the corresponding arbitration outcome coincides with the *Kalai-Smorodinsky solution* outcome (refer to Figure 6).[10] Combining this observation and our fair allocation procedure, we then obtain a mechanism, called the *strategic claim with fair allocation procedure*, which *implements* the Kalai-Smorodinsky solution. Formally, we define the *strategic claim with fair allocation procedure* as follows:

- *Pre-arbitration stage:* Player 1 and player 2 strategically report their claims, $c_1 \in \Delta(X)$ and $c_2 \in \Delta(X)$, respectively, to the arbitrator.

- *Arbitration stage:* The fair allocation procedure is utilized to determine the final outcome.

We obtain the following result.

[10]See also Rong (2012) for this observation.

Theorem 4. *For any given bargaining problem $(d, \Delta(X), u_1, u_2)$, the unique SPE outcome of the strategic claim with fair allocation procedure coincides with the Kalai-Smorodinsky solution outcome.*

Proof: see the appendix. □

Moulin (1984) also designed a mechanism, known as the *auctioning fractions of dictatorship mechanism*,[11] which implements the Kalai-Smorodinsky solution. Our mechanism differs from Moulin (1984) in the sense that the auctioning fractions of dictatorship mechanism has an auction framework, while our mechanism has a simultaneous-offer and alternating-offer framework.

It is worth noticing that our implementation result for the Kalai-Smorodinsky solution also differs from the various support results in the literature regarding the Kalai-Smorodinsky solution (e.g., Trockel (1999), Haake (2000)). In order to support an axiomatic solution, one only needs to design a strategic game (instead of a mechanism) whose equilibrium outcome coincides with the axiomatic solution outcome. A strategic game might depend on players' utilities, and thus the game itself might change as the players' utilities change. In comparison with the support results, designing a mechanism to implement an axiomatic solution is a more challenging work.

Finally, although we assume that $X = \{(x, y) \in R^2 | x \geq 0, y \geq 0, x + y \leq E\}$ and $d = (0, 0)$ throughout the paper, Theorem 4 can be easily extended to the case where X is any compact set such that (i) $S(\Delta(X), u_1, u_2)$ is *d-comprehensive* and (ii) $d \in X$ is the least preferred allocation for both players. A set S is d-comprehensive if for any $(x, y) \in S$, if there exists a (x', y') such that $(u_1(d), u_2(d)) \leq (x', y') \leq (x, y)$, then we must have $(x', y') \in S$.

[11] In the auctioning fractions of dictatorship mechanism, two players simultaneously make bids between 0 and 1. The bidder with the highest bid is the winning bidder. The winning bidder makes an offer to the non-winning bidder. If the offer is accepted, then the offer is implemented. If the offer is rejected, then the non-winning bidder makes a "take it or leave it" offer to the winning bidder with probability p (where p is the bid of the winning bidder), and the disagreement point is implemented with probability $1 - p$.

6 Concluding remarks

This paper considers the scenario where two players are unable to reach agreement with each other, and the two players submit their claims to an arbitrator for arbitration. Both players' claims are legal and verifiable. We propose that a fairness requirement be imposed on the arbitration procedure, which requires that the arbitrator award an outcome that brings the same payoff to the two players whenever the two players' claims are symmetric and the allocation set is symmetric, where both the players' claims and the allocation set are measured in terms of players' utilities. We show that together with other natural axioms, this fairness requirement implies a unique allocation outcome, the fair allocation solution outcome, for any claims problem. We then propose an arbitration procedure, called the fair allocation procedure, which implements the fair allocation solution. Based on the fair allocation procedure, we design a mechanism that implements the Kalai-Smorodinsky solution.

A natural extension of the paper is to consider the n-person case. It is easy to extend Theorem 1 to the n-person case, and the fair allocation solution can be defined as the unique outcome on the Pareto frontier where all n players have identical normalized utility losses. However, since the fair allocation solution outcome varies as players' utilities vary, it seems that in general, the fair allocation solution does not satisfy the consistency condition, which requires that the initial suggestion of a solution for a subgroup of people is the same as the solution to the reduced problem faced by the subgroup.

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Appendix

Proof of Theorem 2:

We can find a positive affine transformation $T(x) = rx + s$ such that $(T \circ v_2)(c_1) = u_2(c_1)$ and $(T \circ v_2)(c_2) = u_2(c_2)$, i.e., $(T \circ k)(u_2(c_1)) = u_2(c_1)$ and $(T \circ k)(u_2(c_2)) = u_2(c_2)$.[12] Then we have $(T \circ v_2)(a^*) = (T \circ k)(u_2(a^*)) \geq u_2(a^*)$, where the equality follows from the fact that $a^* \in X$, and the inequality follows from the facts that $T \circ k$ is concave and strictly increasing, $u_2(c_1) \leq u_2(a^*) \leq u_2(c_2)$, $(T \circ k)(u_2(c_1)) = u_2(c_1)$ and $(T \circ k)(u_2(c_2)) = u_2(c_2)$.

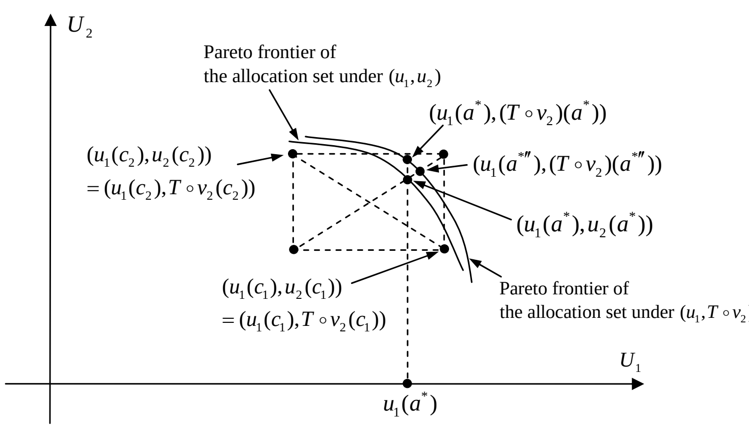


Figure 7: Comparison of two claims problems.

Let $a^{*\prime\prime} = g^*(c_1, c_2, E, u_1, T \circ v_2)$. Since the claims problem (c_1, c_2, E, u_1, v_2) is deterministic, it must be true that $(c_1, c_2, E, u_1, T \circ v_2)$ is also deterministic and thus $a^{*\prime\prime} \in X$. Now, compare the claims problem (c_1, c_2, E, u_1, u_2) and $(c_1, c_2, E, u_1, T \circ v_2)$. Since $(T \circ v_2)(c_1) = u_2(c_1)$ and $(T \circ v_2)(c_2) = u_2(c_2)$, the line that connects the component-wise minimum and component-wise maximum of $(u_1(c_1), u_2(c_1))$ and $(u_1(c_2), u_2(c_2))$ is the same as the line that connects the component-wise minimum and component-wise maximum of $(u_1(c_1), (T \circ v_2)(c_1))$ and $(u_1(c_2), (T \circ v_2)(c_2))$. Noting that $(u_1(a^*), u_2(a^*))$ is on the Pareto frontier of the allocation set $\{(u_1(x), u_2(x)) : x \in X\}$, we must have that $(u_1(a^*), (T \circ v_2)(a^*))$ is on the Pareto frontier of the allocation set $\{(u_1(x), (T \circ v_2)(x)) : x \in X\}$. Refer to Figure 7. It is clear that $(T \circ v_2)(a^{*\prime\prime}) \leq (T \circ v_2)(a^*)$ (using the fact that $(T \circ v_2)(a^*) \geq u_2(a^*)$). That

[12] We use $f \circ g$ to denote the composite function of f and g.

is, $(T \circ k)(u_2(a^{*\prime\prime})) \leq (T \circ k)(u_2(a^*))$. That is, $u_2(a^{*\prime\prime}) \leq u_2(a^*)$ because $T \circ k$ is strictly increasing. So, we have $u_2(a^{*\prime}) \leq u_2(a^*)$ because $a^{*\prime\prime} = a^{*\prime}$ by the scale invariance axiom. □

Proof of Theorem 3:

Let $C = (c_1, c_2, E, u_1, u_2) \in \mathcal{C}$ be a given claims problem. Let $(x_1, y_1) = (u_1(c_1), u_2(c_1))$ and $(x_2, y_2) = (u_1(c_2), u_2(c_2))$. Note that we always use x to represent the payoff obtained by player 1, and y to represent the payoff obtained by player 2.

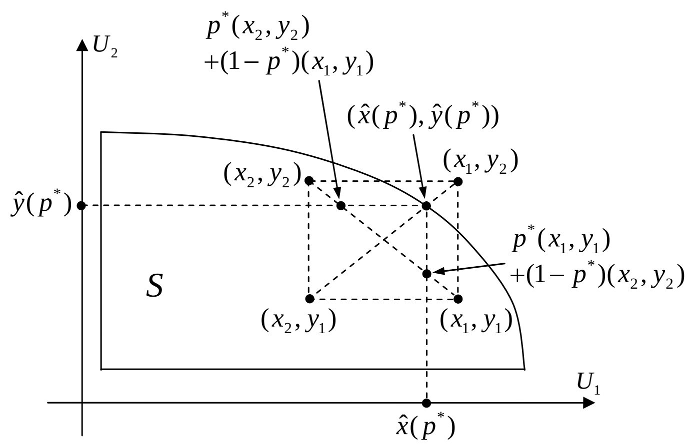


Figure 8: Determination of p^*.

Denote $\hat{x}(p) = px_1 + (1-p)x_2$ and $\hat{y}(p) = py_2 + (1-p)y_1$. Obviously, $\hat{x}(p)$ is player 1's payoff when player 1 requests that the arbitrator end the game, and $\hat{y}(p)$ is player 2's payoff when player 2 requests that the arbitrator end the game. Define p^* as the unique $p \in [0,1]$ such that $(\hat{x}(p), \hat{y}(p))$ lies on the Pareto frontier of bargaining set S (see Figure 8).[13] Let a^* be the lottery in $\Delta(X)$ such that $(u_1(a^*), u_2(a^*)) = (\hat{x}(p^*), \hat{y}(p^*))$. Assume that player 1 is chosen by the arbitrator at stage 0 (the case where player 2 is chosen by the arbitrator is similar). In the remainder of the proof, we show that it is a subgame-perfect equilibrium for player 1 to choose $p = p^*$ and offer a^*, which player 2 accepts.

Since stage 3 will never be reached in equilibrium, we start our analysis from stage 2.

(i) *Stage 2*

[13]The probability p^* exists because $(x_1, y_2) \notin S$ (we have $(x_1, y_2) \notin S$ because c_1 and c_2 are incompatible).

Let $m(p)$ be the lottery in $\Delta(X)$ such that $(u_1(m(p)), u_2(m(p))) = (\hat{x}(p), f(\hat{x}(p)))$, where $f(x)$ is the maximum payoff that player 2 can obtain from the allocation set S given that player 1's demand is x. If the game moves to stage 2, then it is an equilibrium for player 2 to make the offer $m(p)$, which player 1 accepts. Player 2's equilibrium payoff is thus $f(\hat{x}(p))$.

(ii) *Stage 1*

We have the following two cases:

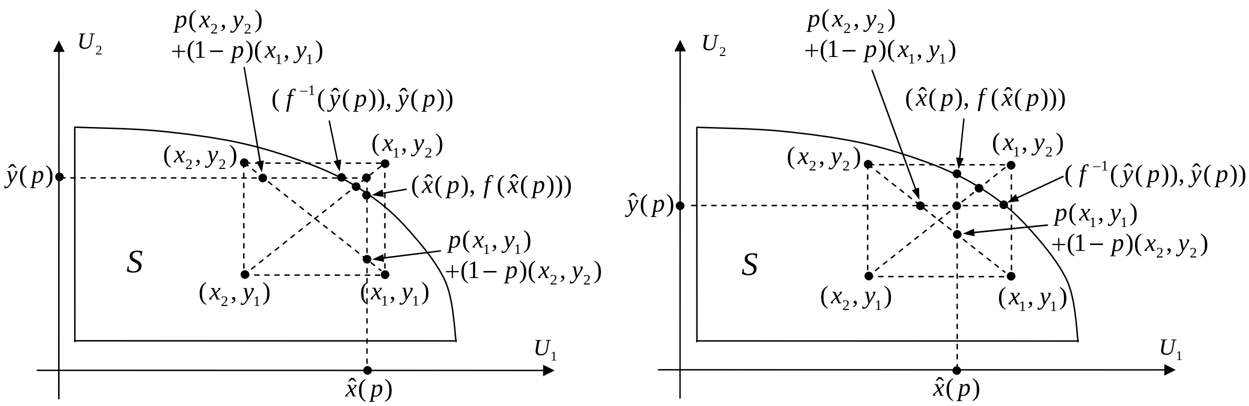
(1) The probability p is such that $(\hat{x}(p), \hat{y}(p)) \notin S$.

Let $c_1' \in \Delta(X)$ be the offer made by player 1 at the beginning of stage 1. Player 2 can either accept the offer or reject it. If player 2 rejects the offer, then player 2 can either request the arbitrator to end the game, obtaining a payoff of $\hat{y}(p)$, or move the game to stage 2, obtaining an equilibrium payoff of $f(\hat{x}(p))$. Since $(\hat{x}(p), \hat{y}(p)) \notin S$, we must have $\hat{y}(p) > f(\hat{x}(p))$ (see the left panel in Figure 9). Thus, player 2's optimal action after rejecting player 1's offer c_1' is to request the arbitrator to end the game, and player 2 obtains a payoff of $\hat{y}(p)$. On the other hand, if player 2 accepts the offer c_1', then his payoff will be $u_2(c_1')$. As a result, player 1's offer c_1' will be accepted by player 2 if and only if $u_2(c_1') \geq \hat{y}(p)$. Let $n(p)$ be the lottery in $\Delta(X)$ such that $(u_1(n(p)), u_2(n(p))) = (f^{-1}(\hat{y}(p)), \hat{y}(p))$. It can be easily verified that player 1's optimal offer is $n(p)$, which will be accepted by player 2 immediately, and player 1 obtains a payoff of $f^{-1}(\hat{y}(p))$.[14]

(2) The probability p is such that $(\hat{x}(p), \hat{y}(p)) \in S$.

Again, let $c_1' \in \Delta(X)$ be the offer made by player 1 at the beginning of stage 1. If player 2 rejects the offer, then player 2 can either request the arbitrator to end the game, obtaining a payoff of $\hat{y}(p)$, or move the game to stage 2, obtaining a payoff of $f(\hat{x}(p))$. Since $(\hat{x}(p), \hat{y}(p)) \in S$, we must have $\hat{y}(p) \leq f(\hat{x}(p))$ (see the right panel in Figure 9). Thus, player 2's optimal action after rejecting player 1's offer c_1' is to move the game to stage 2, and player 2 obtains a payoff of $f(\hat{x}(p))$. On the other hand, if player 2 accepts the offer c_1', then his

[14]If player 1 makes an offer c_1' with $u_2(c_1') < \hat{y}(p)$, then the offer will be rejected by player 2 and player 2 will request the arbitrator to end the game. Player 1's payoff is thus $px_2 + (1-p)x_1$, which is strictly less than $f^{-1}(\hat{y}(p))$ (see the left panel in Figure 9).



The case where $(\hat{x}(p), \hat{y}(p)) \notin S$ The case where $(\hat{x}(p), \hat{y}(p)) \in S$

Figure 9: The choice of the probability p.

payoff will be $u_2(c_1')$. As a result, player 1's offer c_1' will be accepted by player 2 if and only if $u_2(c_1') \geq f(\hat{x}(p))$. It can be easily verified that player 1's optimal offer is $m(p)$,[15] which will be accepted by player 2 immediately, and player 1 obtains a payoff of $\hat{x}(p)$.[16]

Based on the analysis above, player 1's payoff is maximized when $p = p^*$. Thus, it is subgame-perfect equilibrium for player 1 to choose $p = p^*$, and offer a^* at stage 1, and for player 2 to accept a^* immediately (recall that a^* is the lottery such that $(u_1(a^*), u_2(a^*)) = (\hat{x}(p^*), \hat{y}(p^*))$). Observing that for any $p \in [0, 1]$, $(\hat{x}(p), \hat{y}(p))$ lies on the line that connects (x_2, y_1) and (x_1, y_2), the point $(\hat{x}(p^*), \hat{y}(p^*))$ must be the intersection point of the Pareto frontier with the line that connects (x_2, y_1) and (x_1, y_2), i.e., $(\hat{x}(p^*), \hat{y}(p^*))$ must be the fair allocation solution payoff to the claims problem (c_1, c_2, E, u_1, u_2). In other words, the unique SPE payoff of the fair allocation procedure coincides with the fair allocation solution payoff.

□

Proof of Theorem 4:

Recall that b_i is player i's maximal possible utility level from the allocation set S. Let d_i

[15] Recall that $m(p)$ is the lottery in $\Delta(X)$ such that $(u_1(m(p)), u_2(m(p))) = (\hat{x}(p), f(\hat{x}(p)))$.

[16] If player 1 makes an offer c_1' with $u_2(c_1') < f(\hat{x}(p))$, then the offer will be rejected by player 2 and player 2 will move the game to stage 2 in equilibrium. At stage 2, player 2 will make the offer $m(p)$, which player 1 accepts. Player 1's equilibrium payoff is thus $\hat{x}(p)$. As a result, player 1 is indifferent between offering $m(p)$, which will be accepted by player 2, and offering c_1' with $u_2(c_1') < f(\hat{x}(p))$, which will be rejected by player 2. For simplicity, we assume that player 1 makes the offer $m(p)$ in equilibrium. This assumption is made for simplicity and the relaxation of the assumption will not change the equilibrium payoff of the game.

be player i's utility obtained from the disagreement point d, i.e., $d_1 = u_1(d)$ and $d_2 = u_2(d)$.

Arbitration stage

In the proof of Theorem 3, we analyzed the case where player 1's claim c_1 and player 2's claim c_2 are *incompatible*. Since in the strategic claim with fair allocation procedure, we allow players to make claims strategically, it is possible that the claims submit by the two players are *compatible*. The remainder of the proof mainly analyze the equilibrium behavior of players for the case where c_1 and c_2 are compatible, i.e., $(x_1, y_2) \in S$ (recall that $(x_1, y_1) = (u_1(c_1), u_2(c_1))$ and $(x_2, y_2) = (u_1(c_2), u_2(c_2))$). We have the following four cases (for simplicity, we omit the cases where $x_1 = x_2$ or $y_1 = y_2$).[17]

(i) $x_1 < x_2$ and $y_1 < y_2$.

In this case (see Figure 10), it can be readily verified that in equilibrium, (i) if it is player 1 who chooses p, then player 1 will choose $p = 0$ and propose the offer a, where a is such that $(u_1(a), u_2(a)) = (x_2, f(x_2))$, and player 2 accepts a immediately; and (ii) if it is player 2 who chooses p, then player 2 will choose $p = 1$ and propose the offer a, where a is such that $(u_1(a), u_2(a)) = (f^{-1}(y_2), y_2)$, and player 1 accepts a immediately.

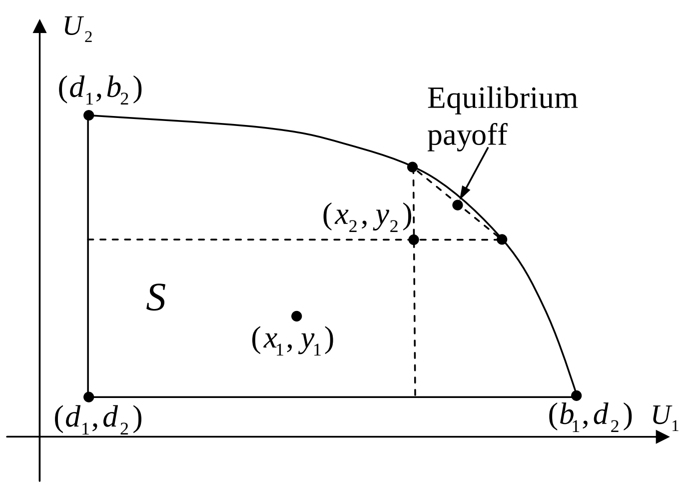


Figure 10: The case where $x_1 < x_2$ and $y_1 < y_2$.

(ii) $x_1 < x_2$ and $y_1 > y_2$.

[17]Given that the analysis of the case of compatible claims is similar to the case of incompatible claims, we will only provide a sketch of the proof below.

In this case, we have two possibilities (see Figure 11): $y_1 \leq f(x_2)$ and $y_1 > f(x_2)$. As regards the first possibility (see also the left panel of Figure 11), it can be readily verified that in equilibrium, (i) if it is player 1 who chooses p, then player 1 will choose $p = 0$ and propose the offer a, where a is such that $(u_1(a), u_2(a)) = (x_2, f(x_2))$, and player 2 accepts a immediately; and (ii) if it is player 2 who chooses p, then player 2 will choose $p = 0$ and propose the offer a, where a is such that $(u_1(a), u_2(a)) = (f^{-1}(y_1), y_1)$, and player 1 accepts a immediately. As regards the second possibility (see also the right panel of Figure 11), it can be verified that in equilibrium if it is player i who chooses p, then player i will choose p^*, where p^* is the unique $p \in [0, 1]$ that satisfies $(\hat{x}(p), \hat{y}(p)) \in P(S)$,[18] and propose the offer a, where a is such that $(u_1(a), u_2(a)) = (\hat{x}(p^*), \hat{y}(p^*))$, and player j accepts a immediately.

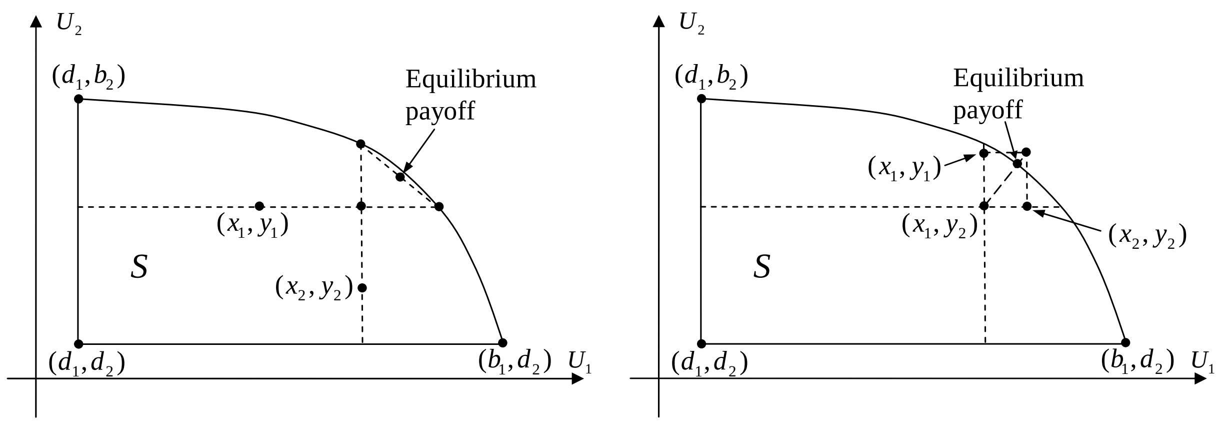


Figure 11: The case where $x_1 < x_2$ and $y_1 > y_2$.

(iii) $x_1 > x_2$ and $y_1 < y_2$.

This case is similar to the first possibility of case (ii) and is omitted.

(iv) $x_1 > x_2$ and $y_1 > y_2$.

This case is similar to case (i) and is omitted.

Pre-arbitration stage

We now return to the pre-arbitration stage. In all the cases mentioned above (including the case where the players' claims are incompatible), player 1 obtains a greater arbitration-stage equilibrium payoff as his pre-arbitration-stage claim (measured in terms of utility

[18] Recall that $\hat{x}(p) = px_1 + (1 - p)x_2$ and $\hat{y}(p) = py_2 + (1 - p)y_1$.

level) $(x_1, y_1) \in S$ moves from the upper-left to the lower-right. Player 2 obtains a greater arbitration-stage equilibrium payoff as his (pre-arbitration-stage) claim (measured in terms of utility level) $(x_2, y_2) \in S$ moves from the lower-right to the upper-left. As a result, in equilibrium, player 1 must claim (measured in terms of utility level) (b_1, d_2) and player 2 must claim (measured in terms of utility level) (d_1, b_2) at the arbitration stage, and the equilibrium outcome of the entire game coincides with the Kalai-Smorodinsky solution outcome. $\square$