Proportional Individual Rationality and the Provision of a Public Good in a Large Economy

Kang Rong*

School of Economics, Shanghai University of Finance and Economics, Shanghai 200433, China

Key Laboratory of Mathematical Economics (SUFE), Ministry of Education, Shanghai 200433, China

Abstract This paper studies the public good provision problem in which a non-excludable public good can be provided and payments can be collected from agents only if the proportion of agents who obtain nonnegative interim expected utilities from the public good provision mechanism weakly exceeds a prespecified ratio α. We call this requirement "α proportional individual rationality." We identify a key threshold such that if α is less than this threshold, then efficiency obtains asymptotically. If α is greater than the threshold, then inefficiency obtains asymptotically. In addition, we obtain the convergence rate of the probability of provision to its efficient/inefficient level. Moreover, as a methodological contribution of this paper, we propose the standard deviation of an agent's interim expected provision as a measure of the agent's influence in a mechanism. We find that as the economy becomes large, an agent's influence in any sequence of anonymous mechanisms converges to zero, and thus any sequence of anonymous feasible mechanisms must converge to a constant mechanism. We obtain uniform bounds for those rates of convergence.

Keywords: Public goods; α proportional individual rationality; Asymptotic efficiency; Asymptotic inefficiency; Influence; Convergence rates.

JEL classification: D82, H41

*E-mail: *rong.kang@shufe.edu.cn*. This paper is based on a chapter of my Ph.D. dissertation at the University of Illinois at Urbana-Champaign. I thank Steven Williams, Bart Taub and Dan Bernhardt for their patient guidance and helpful advice. I thank participants at the UIUC Research Seminar, Midwest Economics Association Meeting, Midwest Economic Theory Meeting, and Washington University's Economics Graduate Conference for helpful comments. I would also like to thank David Martimort for a discussion of the initial idea of the paper. I gratefully acknowledge the financial support from the UIUC Summer Research Fellowship. All errors are mine.

1 Introduction

This paper studies the provision of a (non-excludable) public good. It is well-known that the voluntary provision of public goods may not attain the first-best efficient level due to the free-rider problem. How to design a budget-balanced mechanism[1] that can achieve first-best efficiency is thus a fundamental topic in research in public economics. In the literature, if the interim individual rationality constraint is *not* required for any agent, then first-best efficiency can be achieved (d'Aspremont and Gérard-Varet (1979)). If interim individual rationality is required for *all* agents, then first-best efficiency cannot be achieved (Mailath and Postlewaite (1990)). This paper considers a model that connects these two cases. Specifically, we consider a model in which the public good is provided and payments are collected from agents only if the proportion of agents who obtain nonnegative interim expected utilities from the mechanism weakly exceeds a prespecified *required agreement rate* $\alpha \in [0, 1]$. We call this constraint α *proportional individual rationality.*

The α proportional individual rationality constraint is appropriate in a setting where players vote on the adoption of the mechanism under an α majority rule, and vote sincerely because the chance of being pivotal is negligible. Consequently, we consider the public good provision problem in a *large* finite economy. We are not only interested in whether first-best efficiency can be achieved or not in a large economy, but also interested in the speed at which the efficiency or inefficiency is reached as the economy becomes large. Thus, two basic research questions of this paper are as follows. First, for any given α, can the first-best provision level be achieved in a large economy? Second, how rapidly does the probability of provision approach its efficient or inefficient level as the economy becomes large?

We assume agents' valuations are i.i.d. and the total provision cost of the public good in the n-agent economy is nc, where c is the constant per capita cost of provision. We assume that c is less than the expected value of an agent's valuation. This assumption ensures that

[1] In the case where the budget balance constraint is not required, Groves (1973) and Clarke (1971) showed that first-best efficiency can be achieved by a dominant-strategy mechanism.

the public good *should* be provided with probability 1 in a large economy. Our results are:

(i) If α is less than a threshold α^*, then there exists a sequence of mechanisms that satisfy ex post incentive compatibility, ex post budget balance, and α proportional individual rationality. As n goes to infinity, the probabilities of provision in this sequence of mechanisms approach 1 at an exponential rate.

(ii) If α is greater than α^*, then for *any* sequence of anonymous mechanisms satisfying interim incentive compatibility, ex ante budget balance, and α proportional individual rationality, the probabilities of provision approach 0 as n goes to infinity. In addition, the convergence speed is not slower than $1/n^{\frac{1}{3}}$.

The above results are summarized in Figure 1.

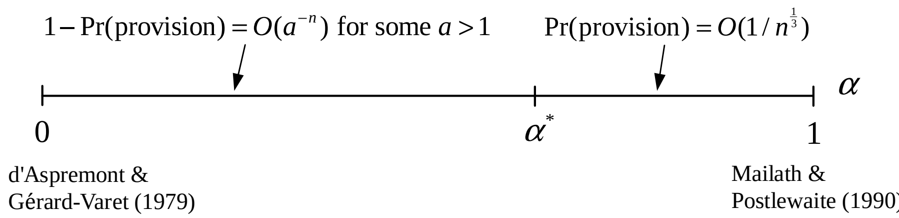

Figure 1: Probability of provision under various α as the economy becomes large.

The threshold α^* equals the probability of an agent's valuation being higher than the per capita cost of provision, i.e., $1 - F(c)$, where F is the distribution function of an agent's valuation. The intuition is as follows. As the economy becomes large, the probability that an agent is pivotal in any anonymous mechanism becomes small. As a result, the agent's expected payment must be nearly constant, independent of his valuation, if he is to report honestly. Ex ante budget balance then requires this constant payment to equal the per capita cost of provision. Thus, in a large economy, the proportion of agents who obtain nonnegative interim expected utilities from the mechanism must be approximately $1 - F(c)$. Whether or not the public good can be provided then depends on whether or not the required agreement rate α is less than $1 - F(c)$.

The results in this paper can be better understood if we compare them with those of Mailath and Postlewaite (1990). They showed that if we require unanimity, then the public good will not be provided in a large economy. This implies that some coercion is necessary if the principal wants the public good to be provided efficiently in a large economy. However, it is not clear from the literature exactly how much coercion is needed. Our results provide an answer to this question. That is, if we want the public good to be provided efficiently in a large economy, then the proportion of agents that are allowed to be hurt in the mechanism should be *at least* $F(c)$.

Furthermore, if we fix a required agreement rate α, and let v^* be the valuation such that $1 - F(v^*) = \alpha$, then efficiency obtains if and only if v^* is greater than the per capita cost c. This result shares the same spirit as the result in Mailath and Postlewaite (1990), who showed that efficiency obtains if and only if the lowest type $\underline{v}$ is greater than c. In other words, our results can be understood in the following manner. For any given α, we can simply chop off the lower end of an agent's distribution of valuation by a proportion of $1 - \alpha$ and obtain v^* as the new lowest type. Whether or not efficiency obtains then depends on whether or not this new lowest type v^* is greater than c.

For the case where α is less than the threshold α^*, the mechanism that we construct is actually quite simple. Each agent is required to report his valuation to the principal simultaneously. The public good is provided if and only if the proportion of agents whose reported valuations exceed the per capita cost is greater than α, and the cost of provision is distributed equally among all agents whenever the public good is provided. Asymptotic efficiency can be attained by this mechanism because as the economy becomes large, the proportion of agents whose valuations exceeds the per capita cost of provision approaches $1 - F(c)$, which is greater than the required agreement rate α by assumption.

For the case where α is greater than the threshold α^*, in order to obtain the convergence rate of the probability of provision toward zero, we need to characterize the limiting behavior of anonymous feasible mechanisms, where a mechanism is feasible if and only if it satisfies all the constraints (i.e., interim incentive compatibility, ex ante budget balance and α proportional individual rationality). Our approach used to characterize the limiting behavior of anonymous feasible mechanisms is very general and may be regarded as a methodological contribution of the paper. In particular, we propose the standard deviation of an agent's interim expected provision[2] as a measure of the agent's *influence* in a mechanism. We find that, as the economy becomes large, an agent's influence in *any* sequence of anonymous mechanisms decreases to zero at a speed not slower than $1/\sqrt{n}$. Based on this result, we find that for any sequence of anonymous feasible mechanisms, an agent's interim expected provision function must converge to a constant function at a speed not slower than $1/\sqrt{n}$, and the agent's interim expected payment function must converge to a constant function at a speed not slower than $1/n^{\frac{1}{3}}$. Roughly speaking, we obtain bounds for the convergence speed of any sequence of anonymous feasible mechanisms toward a "constant mechanism."[3]

Our result about the convergence rate of an agent's influence is similar to the result obtained by Al-Najjar and Smorodinsky (2000), who showed that in any sequence of mechanisms, the average influence of agents converges to zero at a speed not slower than $1/\sqrt{n}$ as the economy becomes large. The main difference between the two results is that the measures of influence are different.[4] In particular, Al-Najjar and Smorodinsky (2000) use the difference between the maximum and the minimum of an agent's interim expected provision to measure the agent's influence, while we use the standard deviation.

In the special case where $\alpha = 1$, Mailath and Postlewaite (1990) showed that the probabilities of provision in any sequence of mechanisms that satisfy interim incentive compatibility, ex ante budget balance, and interim individual rationality converge to zero at a speed not slower than $1/n^{\frac{1}{4}}$. Our convergence rate result improves this bound to $1/n^{\frac{1}{3}}$. It should be noted that Al-Najjar and Smorodinsky (2000) also improved the bound obtained by Mailath

[2]An agent's interim expected provision is the expected value of the provision of the public good conditional on the agent's own report/valuation. It is a function of the agent's own valuation.

[3]By "constant mechanism," we refer to the mechanism where any agent's interim expected provision and payment functions are constant.

[4]Another difference between the two results is that our result is based on the assumptions that agents' valuations are identically distributed and that the mechanism is anonymous, while Al-Najjar and Smorodinsky (2000) do not make such assumptions. However, this difference is not crucial, because our result (in particular, Lemma 2) can be easily generalized to the case where agents' valuations are not identically distributed and the mechanism is non-anonymous.

and Postlewaite (1990) to $1/n^{\frac{1}{3}}$.[5] However, our result is more general in the sense that they rely on the assumption that the distribution of an agent's valuation has an atom at that agent's lowest possible value, while our result does not require this assumption. In addition, Al-Najjar and Smorodinsky (2000) focus on the case where $\alpha = 1$, while our result holds for any $\alpha > 1 - F(c)$.

Our paper also relates to Norman (2004) in the sense that both papers have an asymptotic threshold characterization result for the public good provision. The difference between the two papers is that the public good considered in Norman (2004) is excludable, but it is non-excludable in our paper. In addition, our paper allows some agents to be hurt, while in Norman (2004), no agent can be coerced into participation. Norman (2004) also shows that if agents' valuations are i.i.d., then a fixed fee mechanism is almost constrained-optimal in a large economy. This result shares the same feature with ours because we show that any sequence of anonymous feasible mechanisms must converge to a constant mechanism as the economy becomes large, which implies that a constant mechanism must be asymptotically constrained-optimal. However, Norman (2004) does not characterize the speed of such convergence.

This paper is organized as follows. The next section presents the model. Section 3 discusses the asymptotic efficiency and asymptotic inefficiency results. Section 4 discusses the case where the cost per capita converges to zero as the economy becomes large. Concluding remarks are offered in Section 5.

[5] Al-Najjar and Smorodinsky (2000) showed that when the distribution is continuous on $(\underline{v}, \overline{v}]$ and is discrete at $\underline{v}$ with $P(\underline{v}) \geq \epsilon$ for some $\epsilon > 0$, the probability of provision in any mechanism is smaller than $C_1 \frac{1}{\sqrt{\eta}} \frac{1}{\sqrt{n}} + C_2 \eta$ for any $0 < \eta < \epsilon$ for some constants C_1 and C_2. It can be verified that this result implies that the bound for the probability of provision is $O(1/n^{\frac{1}{3}})$, instead of $O(1/\sqrt{n})$ as stated by Al-Najjar and Smorodinsky (2000) (see Appendix 2 for a detailed proof).

2 Model

Assume that a non-excludable public good can be provided in the quantity of either 0 unit or 1 unit. The cost of providing 1 unit of the public good is nc, where n is the number of agents in the economy, or the size of the economy, and c is a constant.[6] We denote agent i's valuation of the public goods by v_i. Agent i's valuation v_i is known only to agent i. We assume that $v_1, \cdots, v_n$ are independent and identically distributed.[7] The distribution function of v_i is denoted by F, which is common knowledge among all agents and the principal. The support of F is denoted by $[\underline{v}, \overline{v}] \subset R_+$. The density function of F is denoted by f. We assume that f is continuous and strictly positive on $[\underline{v}, \overline{v}]$. Finally, we assume $\underline{v} < c < v^e < \overline{v}$ where $v^e = Ev_i$. The assumption that an agent's expected value of valuation v^e exceeds the per capita cost c is made in order to ensure that in a large economy, the average social benefit of the public good is greater than the average social cost of the public good. As a result, in a large economy, first-best efficiency requires that the public good be provided with probability 1.[8]

We consider direct anonymous mechanisms in this paper. A direct mechanism is a function pair $\{q^n, \{t_i^n\}_{i=1}^n\}$ where $q^n : [\underline{v}, \overline{v}]^n \to \{0, 1\}$ indicates whether or not the public good is provided and $t_i^n : [\underline{v}, \overline{v}]^n \to R$ is the payment collected from agent i. Note that q^n and t_i^n are functions of reported valuations $(\hat{v}_1, \cdots, \hat{v}_n)$ of *all* agents. The anonymity of the mechanism requires q^n and t_i^n to be functions that depend exclusively on the reported valuations and not on the identities of agents.[9] Given agents' reported valuations $(\hat{v}_1, \cdots, \hat{v}_n)$, agent i's ex-

[6]Here, the cost function is such that the cost of provision increases in proportion to the number of agents. This assumption is in the spirit of Mailath and Postlewaite (1990), in which the per capita cost of provision is bounded away from zero.

[7]The i.i.d. setting in the public good provision problem also appeared in Rob (1989) and in Ledyard and Palfrey (2002).

[8]If instead, v^e is less than c, then first-best efficiency requires that the public good be provided with probability 0 in a large economy. As a result, for any given α, first-best efficiency can always be achieved in a large economy by the trivial mechanism where the public good is not provided and payments are not collected from agents for any reported valuations.

[9]Formally, the anonymity of the mechanism requires (i) $q(\hat{v}_1, \cdots, \hat{v}_n) = q(\sigma(\hat{v}_1, \cdots, \hat{v}_n))$ for any permutation $\sigma(\hat{v}_1, \cdots, \hat{v}_n)$ of $(\hat{v}_1, \cdots, \hat{v}_n)$, (ii) $t_i^n(\hat{v}_i, \hat{v}_{-i}) = t_i^n(\hat{v}_i, \sigma(\hat{v}_{-i}))$ for any permutation $\sigma(\hat{v}_{-i})$ of $\hat{v}_{-i}$, and (iii) $t_i^n(\hat{v}_i, \hat{v}_{-i}) = t_j^n(\hat{v}_j, \hat{v}_{-j})$ where $\hat{v}_i = \hat{v}_j$ and $\hat{v}_{-j}$ is a permutation of $\hat{v}_{-i}$.

post utility under the mechanism $\{q^n, \{t_i^n\}_{i=1}^n\}$ is given by $v_i q^n(\hat{v}_1, \cdots, \hat{v}_n) - t_i^n(\hat{v}_1, \cdots, \hat{v}_n)$. Define $\hat{U}_i^n(v_i, \hat{v}_i) = E_{v_{-i}}[v_i q^n(\hat{v}_i, v_{-i}) - t_i^n(\hat{v}_i, v_{-i})]$ and $U_i^n(v_i) = \hat{U}_i^n(v_i, v_i)$. Then $\hat{U}_i^n(v_i, \hat{v}_i)$ represents agent i's interim expected utility when he reports $\hat{v}_i$ and $U_i^n(v_i)$ represents agent i's interim expected utility when he reports truthfully, both conditional on all other agents reporting truthfully. Since the prior distribution of v_i is the same for all i and the mechanism $\{q^n, \{t_i^n\}_{i=1}^n\}$ is anonymous by assumption, the function $U_i^n(v_i)$ (similarly, $\hat{U}_i^n(v_i, \hat{v}_i)$) must be the same for all i and will be denoted by $U^n(v_i)$ ($\hat{U}^n(v_i, \hat{v}_i)$, respectively) whenever there is no confusion.

We impose three constraints on the mechanism. The first constraint is the *interim incentive compatibility* constraint INTIC. With INTIC, the truthful strategy is a Bayesian Nash equilibrium of the mechanism.

$$\text{(INTIC)} : U^n(v_i) \geq \hat{U}^n(v_i, \hat{v}_i) \text{ for all } v_i, \hat{v}_i \in [\underline{v}, \bar{v}] \tag{1}$$

A stronger version of the interim incentive compatibility constraint is the *ex post incentive compatibility* constraint EXPIC.

$$\begin{aligned} \text{(EXPIC)} : v_i q^n(v) - t_i^n(v) \quad &\geq \quad v_i q^n(\hat{v}_i, v_{-i}) - t_i^n(\hat{v}_i, v_{-i}) \text{ for any } v_i, \hat{v}_i \in [\underline{v}, \bar{v}] \\ &\text{and } v_{-i} \in [\underline{v}, \bar{v}]^{n-1} \end{aligned} \tag{2}$$

For a direct mechanism, the truthful strategy is a weakly dominant strategy of the mechanism if and only if the mechanism satisfies EXPIC. Obviously, if a mechanism satisfies EXPIC, then it must satisfy INTIC.

The second constraint is the *ex ante budget balance* constraint EXABB.

$$\text{(EXABB)} : E\{\frac{\sum t_i^n(v)}{n} - cq^n(v)\} \geq 0 \tag{3}$$

A stronger version of the ex ante budget balance constraint is the *ex post budget bal-*

ance constraint EXPBB, which requires that the mechanism be budget-balanced for any *realization* of valuations.[10]

$$(\text{EXPBB}) : \frac{\sum t_i^n(v)}{n} - cq^n(v) \geq 0 \text{ for all } v \in [\underline{v}, \bar{v}]^n \tag{4}$$

The last constraint is the α *proportional individual rationality* constraint. This constraint reflects the requirement that the public good can be provided and payments can be collected from agents only if the proportion of agents who obtain nonnegative interim expected utilities from the mechanism is at least α. In other words, the α-PIR constraint reflects the requirement that a proportion α of the agents need to approve the mechanism in order for the mechanism to be enforced.[11]

With the α-PIR constraint, as long as the mechanism is approved, all agents are forced to participate regardless of their IR constraints. However, it should be noted that even if the mechanism is approved, it does not mean the public good will be provided. That is, agents are voting on the public good provision mechanism, instead on whether to provide the public good.

We will introduce some notation before we formally define the α proportional individual rationality constraint. Define the *agreement set* $\tilde{V}^n$ as the set of valuations for which an agent's interim expected utility is nonnegative, i.e., $\tilde{V}^n = \{v_i \in [\underline{v}, \bar{v}] | U^n(v_i) \geq 0\}$. Define $r^n(v) = \frac{\sum_{i=1}^n \mathbf{1}_{\{v_i \in \tilde{V}^n\}}}{n}$ where $\mathbf{1}_{\{\bullet\}}$ represents the indicator function. Then, $r^n(v)$ is the proportion of agents who obtain nonnegative interim expected utilities from the mechanism. Now, the α *proportional individual rationality* constraint is:

[10]Although EXPBB seems to be a stronger requirement than EXABB, it is well known in the literature that EXABB and EXPBB are actually equivalent in the sense that the for any mechanism satisfying EXABB, there exists a mechanism that satisfies EXPBB and yields the same interim expected utility for any agent (see, e.g., Börgers and Norman (2009)).

[11]We can imagine that there exists a voting procedure before the implementation of the public good provision mechanism. That is, a public good provision mechanism is proposed by a principal, and a group of agents then vote on the mechanism, using the α majority rule. We assume that agents vote sincerely so that an agent vote for the mechanism if and only if the agent obtains a nonnegative expected utility from the mechanism. The sincere-voting assumption is appropriate for our model because we consider a large economy, where the probability that any agent is pivotal is small.

$$(\alpha\text{-PIR}) : q^n(v) = 0 \text{ and } t_i^n(v) = 0 \text{ (for all } i) \text{ if } r^n(v) < \alpha.$$

In this paper, α is exogenously fixed and can be any number between 0 and 1. If α=0, then the α proportional individual rationality constraint is automatically satisfied by any mechanism. If α=1, then the α proportional individual rationality constraint is satisfied if and only if the interim expected utility is nonnegative for all agents, i.e., if and only if the interim individual rationality constraint is satisfied for all agents.[12]

The α-PIR constraint actually describes what is happening if the proportion of the agents who approve the mechanism is less than the required agreement rate α. One may think that the α-PIR constraint should be defined in a manner such that a mechanism satisfies the constraint if at least a proportion α of the agents are better off than the outside option (i.e., a proportion α of the agents obtain nonnegative interim expected utilities). This definition is problematic for the following reason. A mechanism is designed by the principal at the ex ante stage. For a given mechanism, whether there exists a proportion α of the agents who are better off than the outside option depends on the distribution of the *realized* valuations of all agents. In other words, at the ex ante stage, when the principal designs the mechanism, the principal cannot require the mechanism to be such that at least α proportion of the agents are better off than the outside option because the principal does not know the realized valuations of the agents.

The following lemma will be used to simplify the α-PIR constraint. Define $q_i^n(v_i) = E_{v_{-i}} q^n(v)$ as agent i's *interim expected provision* and $t_i^n(v_i) = E_{v_{-i}} t_i^n(v)$ as agent i's *interim expected payment*. We have:

Lemma 1. *If the mechanism* $\{q^n, \{t_i^n\}_{i=1}^n\}$ *satisfies INTIC, then:*

(i) $q_i^n(v_i)$, $t_i^n(v_i)$ *and* $U^n(v_i)$ *are nondecreasing on* $[\underline{v}, \bar{v}]$*;*

[12] The "if" part is obvious. To see the "only if" part, if there is a $v_i^* \in [\underline{v}, \bar{v}]$ such that $U_i(v_i^*) < 0$, then we must have $r^n(v_i^*, v_{-i}) < 1$ for all v_{-i}, which implies $q^n(v_i^*, v_{-i}) = 0$ and $t_i^n(v_i^*, v_{-i}) = 0$ for all v_{-i} by the α proportional individual rationality constraint. Thus, $U_i(v_i^*)$ must equal zero, which is a contradiction with $U_i(v_i^*) < 0$.

(ii) $U^n(v_i)$ *is continuous on* $[\underline{v}, \bar{v}]$.

Lemma 1 is well-known in the mechanism design literature (see, e.g., Myerson (1981)). Lemma 1 (i) essentially says that in order to make an agent truthfully reveal his information, the agent's interim expected provision function, interim expected payment function and interim expected utility function must be nondecreasing in the agent's valuation. Lemma 1 (ii) implies that $U^n(v_i)$ is either (i) equal to zero at some point in $[\underline{v}, \overline{v}]$, (ii) greater than zero for all $v_i \in [\underline{v}, \overline{v}]$, or (iii) less than zero for all $v_i \in [\underline{v}, \overline{v}]$. We define $\hat{v}^n$ as follows:

$$\hat{v}^n = \begin{cases} \min\{v_i | v_i \in [\underline{v}, \bar{v}] \text{ and } U^n(v_i) = 0\} & if \quad U^n(v_i) = 0 \text{ for some } v_i \in [\underline{v}, \overline{v}]; \\ \underline{v} & if \quad U^n(v_i) > 0 \text{ for all } v_i \in [\underline{v}, \overline{v}]; \\ \infty & if \quad U^n(v_i) < 0 \text{ for all } v_i \in [\underline{v}, \overline{v}]. \end{cases}$$

Using the definition of $\hat{v}^n$ and the fact that $U^n(v_i)$ is nondecreasing in v_i, we have

$$\tilde{V}^n = \{v_i \in [\underline{v}, \bar{v}] | U^n(v_i) \geq 0\} = \{v_i \in [\underline{v}, \bar{v}] | v_i \geq \hat{v}^n\}.$$

Thus,

$$r^n(v) = \frac{\sum_{i=1}^n \mathbf{1}_{\{v_i \in \tilde{V}^n\}}}{n} = \frac{\sum_{i=1}^n \mathbf{1}_{\{v_i \geq \hat{v}^n\}}}{n}.$$

The α-PIR constraint can be rewritten as:

$$(\alpha\text{-PIR}) : q^n(v) = 0 \text{ and } t_i^n(v) = 0 \text{ (for all } i) \text{ if } \frac{\sum_{i=1}^n \mathbf{1}_{\{v_i \geq \hat{v}^n\}}}{n} < \alpha. \tag{5}$$

We now define the first-best mechanism. The first-best mechanism $\{q^{FB(n)}, \{t_i^{FB(n)}\}_{i=1}^n\}$ is any ex ante budget-balanced mechanism that satisfies the Lindahl-Samuelson provision rule:

$$q^{FB(n)} = \begin{cases} 1 & \text{if } \frac{\sum v_i}{n} \geq c; \\ 0 & \text{otherwise.} \end{cases}$$

By the weak law of large numbers, as $n \to \infty$, the per capita benefit $\dfrac{\sum v_i}{n}$ of providing the public good in the first-best mechanism approaches v^e, which is greater than the per capita cost c by assumption. Thus, the probability that the public good will be provided in the first-best mechanism must approach 1 as n goes to infinity, i.e., $P(q^{FB(n)}(v) = 1) \to 1$ as $n \to \infty$.

Finally, we define the *per capita welfare* of the mechanism $\{q^n, \{t_i^n\}_{i=1}^n\}$ as the per capita expected value of the net benefit of the mechanism, i.e., $W(q^n) = \dfrac{1}{n}E\left\{(\sum v_i - nc)q^n(v)\right\} = E\left\{(\dfrac{\sum v_i}{n} - c)q^n(v)\right\}$. Obviously, the per capita welfare of the first-best mechanism, $W(q^{FB(n)})$, approaches $v^e - c$ as $n \to \infty$.

3 Analysis

3.1 Asymptotic efficiency result

In this section, we will explore whether first-best efficiency can be achieved by mechanisms that are constrained by INTIC, EXABB and α-PIR.

In this subsection, we will show that, if $\alpha < 1 - F(c)$, then first-best efficiency can be achieved asymptotically by the sequence of mechanisms $\{M_\alpha^n\}_{n=1}^\infty$, where $M_\alpha^n = \{\tilde{q}^{n,\alpha}, \{\tilde{t}_i^{n,\alpha}\}_{i=1}^n\}$ is constructed so that $\tilde{q}^{n,\alpha}(v) = 1$ and $\tilde{t}_i^{n,\alpha}(v) = c$ for any i if $\dfrac{\sum_{i=1}^n \mathbf{1}_{\{v_i \geq c\}}}{n} \geq \alpha$, and $\tilde{q}^{n,\alpha}(v) = 0$ and $\tilde{t}_i^{n,\alpha}(v) = 0$ for any i if $\dfrac{\sum_{i=1}^n \mathbf{1}_{\{v_i \geq c\}}}{n} < \alpha$. Simply speaking, M_α^n requires that, if there exist at least α proportion of agents whose reported valuations of the public good exceed c, then the public good will be provided and the cost will be distributed equally among all agents. Otherwise, the public good will not be provided and no payment is collected from any agent. This mechanism is equivalent to the voting mechanism in which agents can vote on the provision of the public good and the public good will be provided (with taxes equally distributed among agents) if and only if the proportion of agents who vote for the provision of the public good is greater than α. We thus call M_α^n the *α-referendum* of the

n-agent economy.[13] We have the following asymptotic efficiency result for the α-referendum.

Theorem 1.

(i) M^n_α satisfies EXPIC, EXPBB, and α-PIR;

(ii) If $\alpha < 1 - F(c)$, then the probability of provision in M^n_α converges to 1 at a speed not slower than $2exp\{-2n(\beta - \alpha)^2\}$, where $\beta = 1 - F(c) > \alpha$.

Proof: See Appendix 1. □

Theorem 1 (i) is obvious. M^n_α is ex post budget balanced by construction. It satisfies EXPIC constraint because any agent with valuation above c has the incentive to report truthfully to increase the probability of provision and any agent with valuation below c has the incentive to report truthfully to decrease the probability of provision; the truthful strategy is actually a dominant strategy for each agent. Finally, noting that the interim expected utility under M^n_α equals to zero at $v_i = c$, it can be verified that M^n_α satisfies the α-PIR constraint.

The intuition of the proof for Theorem 1 (ii) is as follows. In the α-referendum M^n_α, the public good will be provided if and only if the proportion of agents whose valuations exceed c is greater than α. By the weak law of large numbers, as $n \to \infty$, the proportion of agents whose valuations exceed c approaches $P(v_i \geq c) = 1 - F(c)$, which is greater than α by assumption. Hence, the probability of the public good being provided under M^n_α approaches 1 as n goes to infinity. Using Hoeffding's inequality for sums of bounded random variables, we can show that the convergence of $P(\tilde{q}^{n,\alpha}(v) = 1)$ toward 1 is at the exponential rate.

We have the following result regarding the per capita welfare of the α-referendum.

Corollary 1. *Assume that $\alpha < 1 - F(c)$, then the difference between the per capita welfare*

[13] The α-referendum defined in this paper is equivalent with the j^*-referendum defined in Ledyard and Palfrey (2002) with $\alpha = \frac{j^*}{n}$. While Ledyard and Palfrey (2002) focused on whether or not *interim efficiency* can be achieved by referenda asymptotically, we focus on whether or not *ex-post efficiency* can be achieved by referenda asymptotically. In addition, there is no individual rationality constraint in the model of Ledyard and Palfrey (2002), while we require α proportional individual rationality.

of the first-best mechanism and the per capita welfare of the α-referendum (i.e., $W(q^{FB(n)}) - W(\tilde{q}^{n,\alpha})$), approaches 0 as $n \to \infty$.

Remark: Assume that $\alpha < 1 - F(c)$ and fix any $\alpha' \in [\alpha, 1 - F(c))$, then it can be verified that the per capita welfare of the α'-referendum also approaches the per capita welfare of the first-best mechanism (in addition, the α'-referendum satisfies EXPIC, EXPBB, and α-PIR). However, the gap between the per capita welfare of the α'-referendum and the per capita welfare of the first-best mechanism is usually the smallest when $\alpha' = \alpha$.

Corollary 1 implies that the per capita welfare losses from using α-referenda instead of first-best mechanisms vanish as n increases. The performance of α-referenda are illustrated in the following example.

Example 1: Assume that v_i is uniformly distributed on $[0, 1]$, and that $c = 0.4$ and $\alpha = 0.5$.

Table 1 lists the probabilities that the public good will be provided in 0.5-referenda as n increases from 5 to 320. It also lists the per capita welfare of 0.5-referenda, the per capita welfare of first-best mechanisms and the per capita welfare losses (i.e., $1 - \frac{W(\tilde{q}^{n,0.5})}{W(q^{FB(n)})}$) from using 0.5-referenda instead of first-best mechanisms.

Table 1: Probability of provision and relative efficiency of 0.5-referenda

n	$1 - P(\tilde{q}^{n,0.5} = 1)$	$W(\tilde{q}^{n,0.5})$	$W(q^{FB(n)})$	$1 - \frac{W(\tilde{q}^{n,0.5})}{W(q^{FB(n)})}$
5	0.3169	0.1097	0.1165	0.0581
10	0.1662	0.1034	0.1062	0.0266
20	0.1273	0.0990	0.1016	0.0264
40	0.0747	0.0981	0.1002	0.0209
80	0.0267	0.0990	0.1001	0.0095
160	0.0041	0.0997	0.1000	0.0021
320	0.0001	0.0999	0.0999	0.0001

The threshold in our example is $1 - F(c) = 0.6$. Since $\alpha = 0.5$ is less than the threshold, first-best efficiency can be achieved asymptotically. In particular, Table 1 shows that as n increases, $1 - P(\tilde{q}^{n,0.5})$ approaches 0 rapidly. It also shows that the per capita welfare loss from the 0.5-referendum vanishes rapidly as n increases. Actually, when the size of the economy is greater than 80, the per capita welfare loss is less than 1%.

3.2 Asymptotic inefficiency result

This subsection considers the case where the required agreement rate is large. We will show that, if $\alpha > 1 - F(c)$, then the probabilities that the public good is provided in any sequence of anonymous mechanisms satisfying INTIC, EXABB and α-PIR approach 0 as $n \to \infty$ and the convergence speed is not slower than $1/n^{\frac{1}{3}}$.

The next lemma states that the variance of any agent's interim expected provision in an n-agent anonymous economy must converge to 0 at a speed not slower than $1/n$ as $n \to \infty$. Let $Var(q_i^n(v_i)) = \int_{[\underline{v},\overline{v}]}[q_i^n(v_i) - Eq_i^n(v_i)]^2 dF(v_i)$. We have:

Lemma 2. *For any sequence of anonymous mechanisms* $\{\{q^n, \{t_i^n\}_{i=1}^n\}\}_{n=1}^{\infty}$*, we have* $Var(q_i^n(v_i)) = O(1/n)$*.*

Proof: See Appendix 1. □

Although Lemma 2 is not necessary if our purpose is just to obtain the asymptotic inefficiency result, it is essential for us to establish the convergence rate of the probability of provision toward zero.

The intuition of Lemma 2 is as follows. Using the assumption that $v_1, \cdots, v_n$ are independent, one can show that $\sum_i Var(q_i^n(v_i)) \leq Var(q^n(v))$. Since $Var(q^n(v)) \leq 1$ and $v_1, \cdots, v_n$ are identically distributed, we thus have $Var(q_i^n(v_i)) = O(1/n)$.

We define agent i's *influence* relative to the mechanism $\{q^n, \{t_i^n\}_{i=1}^n\}$ by $\sqrt{Var(q_i^n(v_i))}$. $\sqrt{Var(q_i^n(v_i))}$ measures the impact on the interim expected provision as agent i's reported valuation changes. A small $\sqrt{Var(q_i^n(v_i))}$ indicates a small impact. If $\sqrt{Var(q_i^n(v_i))}$ is zero, then $q_i^n(v_i)$ equals a constant function almost surely, which implies that agent i essentially has no influence in the mechanism $\{q^n, \{t_i^n\}_{i=1}^n\}$. Lemma 2 says that as n goes to infinity, an agent's influence in any sequence of anonymous mechanisms decreases to zero at a speed not slower than $1/\sqrt{n}$.

The next lemma is a direct result of Lemma 2. It says that for any $\tilde{v} \in (\underline{v}, \bar{v})$, the differences between $q_i^n(\tilde{v})$ and $Eq_i^n(v_i)$ under any sequence of interim incentive compatible anonymous mechanisms vanish as $n \to \infty$ and are $O(1/\sqrt{n})$. This implies that for any sequence of interim incentive compatible anonymous mechanisms, as the economy becomes large, an agent's interim expected provision function must converge to a constant function, and the convergence speed is not slower than $1/\sqrt{n}$.

For any given $\tilde{v} \in (\underline{v}, \bar{v})$, define $m(\tilde{v}) = \min(P(v_i \geq \tilde{v}), P(v_i \leq \tilde{v}))$ (notice that $m(\tilde{v}) > 0$ because the probability density function of v_i is positive on $[\underline{v}, \overline{v}]$). We have:

Lemma 3. *Let $\{\{q^n, \{t_i^n\}_{i=1}^n\}\}_{n=1}^\infty$ be any sequence of anonymous mechanisms where, for each n, $\{q^n, \{t_i^n\}_{i=1}^n\}$ satisfies INTIC. Then, for any given $\tilde{v} \in (\underline{v}, \bar{v})$, we have:*

$$|q_i^n(\tilde{v}) - Eq_i^n(v_i)| \leq \frac{1}{2\sqrt{m(\tilde{v})n}}.$$

Proof: See Appendix 1. □

The intuition for the proof of Lemma 3 can be explained as follows. Lemma 2 insures that the variance of $q_i^n(v_i)$ approaches zero as n goes to infinity. Hence, as n becomes large, $q_i^n(v_i)$ becomes "flat." This implies that for *almost* every point $v_i \in (\underline{v}, \bar{v})$, $q_i^n(v_i)$ converges to $E(q_i^n(v_i))$. That is, the set of points in $(\underline{v}, \bar{v})$ for which the convergence does not hold has measure zero. Since $q_i^n(v_i)$ is nondecreasing in $(\underline{v}, \bar{v})$ (which is true for any mechanism that satisfies INTIC), we can show that actually for *all* of the points in $(\underline{v}, \bar{v})$, we have $q_i^n(v_i)$ converging to $E(q_i^n(v_i))$.[14] Using the result in Lemma 2 and Chebyshev's inequality, we can show that the convergence speed of $q_i^n(v_i)$ to $E(q_i^n(v_i))$ is not slower than $1/\sqrt{n}$.

There is a natural connection between Lemma 3 and the result about influence in Al-

[14]The reason here is that if there is an "outlier" point at which the convergence does not hold, then given that $q_i^n(v_i)$ is nondecreasing, there must exists a set of points with positive measure for which the convergence does not hold.

Najjar and Smorodinsky (2000). To see this, assume that v_i follows a discrete distribution on V, where V is a finite set and each element in V has a probability of at least ϵ. Lemma 3 then implies that $|q_i^n(v_i) - q_i^n(v_i')| \leq \frac{1}{\sqrt{\epsilon n}}$ for any v_i, $v_i' \in V$.[15] This implies that $\max_{v_i, v_i' \in V}(q_i^n(v_i) - q_i^n(v_i')) \leq \frac{1}{\sqrt{\epsilon n}}$. On the other hand, Al-Najjar and Smorodinsky (2000) propose $\max_{v_i, v_i' \in V}(q_i^n(v_i) - q_i^n(v_i'))$ as the measure of the influence of an agent, and show that an agent's influence is roughly bounded by $\frac{1}{\sqrt{\pi}\sqrt{\epsilon n}}$ when n is large. So, roughly speaking, for the case where the agents' valuations are discrete, our measure of influence can be transformed to the measure of influence in Al-Najjar and Smorodinsky (2000), and the bound obtained by our paper and that obtained by Al-Najjar and Smorodinsky (2000) differ only by a constant.

In the case where the distributions of agent's valuations are continuous, the comparison between our measure of influence and that of Al-Najjar and Smorodinsky (2000) is less straightforward. The reason is that in Al-Najjar and Smorodinsky (2000), it is problematic to directly use $\max_{v_i, v_i' \in V}(q_i^n(v_i) - q_i^n(v_i'))$ as a measure of influence in the continuous case, because in doing so, the influence of an agent may then be very large even if $q_i^n(v_i)$ is a constant on a set with a measure of one. Al-Najjar and Smorodinsky (2000) thus propose to use $\inf_{\{A \subset [\underline{v}, \overline{v}]: P(A) < \epsilon\}} \sup_{\{v_i \in A\}} q_i^n(v_i) - \sup_{\{A \subset [\underline{v}, \overline{v}]: P(A) < \epsilon\}} \inf_{\{v_i \in A\}} q_i^n(v_i)$ as a measure of influence. However, this measure of influence depends on an exogenous parameter ϵ, while our measure of influence ($\sqrt{Var(q_i^n(v_i))}$) does not depend on any exogenous parameter. We thus cannot directly compare those two measures in the continuous case.

A good measure of an agent's influence in a mechanism should rely only on the mechanism, and not on any exogenous parameter. It is in this sense that our measure of influence seems to be more suitable than that of Al-Najjar and Smorodinsky (2000) for the case where agents' valuations are continuous.

[15] Although we assume that v_i follows a continuous distribution, the results in Lemma 2 and Lemma 3 also hold for the case where v_i is discrete.

Now, we define

$$\mathcal{M}^n = \{\{q^n, \{t_i^n\}_{i=1}^n\} | \{q^n, \{t_i^n\}_{i=1}^n\} \text{ satisfies INTIC, EXABB,}$$
$$\text{and } \alpha\text{-PIR in the } n\text{-agent economy}\}.$$

We call any mechanism in $\mathcal{M}^n$ a *feasible* mechanism of the n-agent economy. The next lemma characterizes the convergence speed at which the interim expected payment functions in any sequence of anonymous feasible mechanisms approach a constant function. Recall that $m(\tilde{v}) = \min(P(v_i \geq \tilde{v}), P(v_i \leq \tilde{v})) > 0$ for any $\tilde{v} \in (\underline{v}, \overline{v})$. We have:

Lemma 4. *Let $\{\{q^n, \{t_i^n\}_{i=1}^n\}\}_{n=1}^\infty$ be any sequence of anonymous mechanisms where $\{q^n, \{t_i^n\}_{i=1}^n\} \in \mathcal{M}^n$ for each n. Then, for any $\tilde{v} \in (\underline{v}, \overline{v})$ and any $0 < \eta < m(\tilde{v})$, we have $|t_i^n(\tilde{v}) - Et_i^n(v_i)| \leq \overline{v}\dfrac{1}{\sqrt{\eta n}} + 2\overline{v}\eta$.*

Proof: See Appendix 1. □

Notice that $m(\tilde{v})$ does not depend on n. The bound obtained in Lemma 4 is the tightest when $\eta = 4^{-\frac{2}{3}} n^{-\frac{1}{3}}$. This implies that the bound obtained in Lemma 4 is on the order of $1/n^{\frac{1}{3}}$ (instead of $1/\sqrt{n}$). Thus, an agent's interim expected payment function converges to a constant function at a speed not slower than $1/n^{\frac{1}{3}}$ as $n \to \infty$.

The intuition of the proof of Lemma 4 is as follows. The constraint INTIC on the mechanism $\{q^n, \{t_i^n\}_{i=1}^n\}$ implies that $v_i'(q_i^n(v_i) - q_i^n(v_i')) \leq t_i^n(v_i) - t_i^n(v_i') \leq v_i(q_i^n(v_i) - q_i^n(v_i'))$ for any v_i, $v_i' \in [\underline{v}, \overline{v}]$. That is, the variation of $t_i^n(v_i)$ is bounded by the variation of $q_i^n(v_i)$. By Lemma 3, the interim expected provision function q_i^n approaches a constant function as n goes to infinity. Thus, the interim expected payment function t_i^n must also converge to a constant function as the economy becomes large. The bound $\overline{v}\dfrac{1}{\sqrt{\eta n}} + 2\overline{v}\eta$ in Lemma 4 can be explained as follows. Fix $\tilde{v} \in (\underline{v}, \overline{v})$, and consider the difference between $t_i^n(\tilde{v})$ and $t_i^n(v_i)$ for any given $v_i \in [\underline{v}, \overline{v}]$. By Lemma 3, $|q_i^n(\tilde{v}) - q_i^n(v_i)|$ is bounded by $\dfrac{1}{\sqrt{\eta n}}$, where $\eta = \min\{m(\tilde{v}), m(v_i)\}$. So, using INTIC, $|t_i^n(\tilde{v}) - t_i^n(v_i)|$ is bounded by $\overline{v}\dfrac{1}{\sqrt{\eta n}}$. Notice that the threshold $\min\{m(\tilde{v}), m(v_i)\}$ goes to zero if v_i is close to $\underline{v}$ or $\overline{v}$. However, in the case where v_i is close to $\underline{v}$ or $\overline{v}$, we can use the fact that $|t_i^n(v_i)|$ is bounded by $\overline{v}$ to get a bound on $|t_i^n(\tilde{v}_i) - t_i^n(v_i)|$. Combining the above two bounds and taking expectation of $t_i^n(\tilde{v}) - t_i^n(v_i)$ over v_i, we then have that $|t_i^n(\tilde{v}) - Et_i^n(v_i)|$ is bounded by $\overline{v}\frac{1}{\sqrt{\eta n}} + 2\overline{v}\eta$.

The following theorem argues that the probabilities that the public good is provided in any sequence of anonymous feasible mechanisms approach 0 at a speed not slower than $1/n^{\frac{1}{3}}$. Let v_α be such that $P(v_i \geq v_\alpha) = \alpha$. Note that $\alpha > 1 - F(c)$ implies $v_\alpha < c$. We have:

Theorem 2. *Assume that $\alpha > 1 - F(c)$. Let $\{\{q^n, \{t_i^n\}_{i=1}^n\}\}_{n=1}^\infty$ be any sequence of anonymous mechanisms where $\{q^n, \{t_i^n\}_{i=1}^n\} \in \mathcal{M}^n$ for each n, then we have $P(q^n(v) = 1) = O(1/n^{\frac{1}{3}})$.*

Proof: Let $\{\{q^n, \{t_i^n\}_{i=1}^n\}\}_{n=1}^\infty$ be a given sequence of anonymous mechanisms where $\{q^n, \{t_i^n\}_{i=1}^n\} \in \mathcal{M}^n$ for each n. Fix an $\epsilon \in (0, c - v_\alpha)$.[16] For any given n, we have the following two cases.

(i) $\hat{v}^n < v_\alpha + \epsilon$.

In this case, since $\hat{v}^n < v_\alpha + \epsilon < \infty$, we must have $U^n(\hat{v}^n) \geq 0$ by the definition of $\hat{v}^n$. The incentive compatibility constraint implies that $U^n(v_i)$ is nondecreasing in v_i, so $U^n(v_\alpha + \epsilon) \geq U^n(\hat{v}^n) \geq 0$. That is:

$$(v_\alpha + \epsilon)q_i^n(v_\alpha + \epsilon) \geq t_i^n(v_\alpha + \epsilon). \tag{6}$$

Since $\{q_i^n(v_i), t_i^n(v_i)\}$ satisfies INTIC and $\underline{v} < v_\alpha + \epsilon < c < \overline{v}$, then by Lemma 3, we have:

$$q_i^n(v_\alpha + \epsilon) \leq Eq_i^n(v_i) + \frac{1}{2\sqrt{m(v_\alpha + \epsilon)n}} \tag{7}$$

By Lemma 4, for any $0 < \eta < m(v_\alpha + \epsilon)$ (recall that $m(v_\alpha + \epsilon) = \min\{P(v_i \leq v_\alpha + \epsilon), P(v_i \geq v_\alpha + \epsilon)\}$), we have:

[16]The precise choice of ϵ will not affect the convergence rate we obtain below.

$$t_i^n(v_\alpha+\epsilon) \geq Et_i^n(v_i) - \bar{v}\frac{1}{\sqrt{\eta n}} - 2\bar{v}\eta \tag{8}$$

(6) (7) and (8) then imply an upper bound for $Et_i^n(v_i)$, that is:

$$Et_i^n(v_i) \leq (v_\alpha+\epsilon)Eq_i^n(v_i) + \frac{v_\alpha+\epsilon}{2\sqrt{m(v_\alpha+\epsilon)n}} + \bar{v}\frac{1}{\sqrt{\eta n}} + 2\bar{v}\eta \tag{9}$$

Now, by the budget balance constraint, we can get a lower bound for $Et_i^n(v_i)$, that is:

$$Et_i^n(v_i) \geq cEq_i^n(v_i). \tag{10}$$

Inequality (9) and inequality (10) then imply:

$$Eq_i^n(v_i) \leq \frac{1}{c-(v_\alpha+\epsilon)}\left(\frac{v_\alpha+\epsilon}{2\sqrt{m(v_\alpha+\epsilon)}}\frac{1}{\sqrt{n}} + \bar{v}\frac{1}{\sqrt{\eta}}\frac{1}{\sqrt{n}} + 2\bar{v}\eta\right). \tag{11}$$

Since $Eq_i^n(v_i) = Eq^n(v) = P(q^n(v)=1)$, then we have the following inequality for any $0<\eta<m(v_\alpha+\epsilon)$:

$$P(q^n(v)=1) \leq \frac{1}{c-(v_\alpha+\epsilon)}\left(\frac{v_\alpha+\epsilon}{2\sqrt{m(v_\alpha+\epsilon)}}\frac{1}{\sqrt{n}} + \bar{v}\frac{1}{\sqrt{\eta}}\frac{1}{\sqrt{n}} + 2\bar{v}\eta\right). \tag{12}$$

Notice that inequality (12) holds for any sufficiently small η. It can be verified that the bound obtained in (12) is the tightest when $\eta = 4^{-\frac{2}{3}}n^{-\frac{1}{3}}$. This implies that the bound obtained in (12) is on the order of $1/n^{\frac{1}{3}}$.

(ii) $\hat{v}^n \geq v_\alpha+\epsilon$.

Define $\gamma^n(v) = \frac{\sum_{i=1}^n \mathbf{1}_{\{v_i\geq v_\alpha+\epsilon\}}}{n}$. Note $r^n(v) = \frac{\sum_{i=1}^n \mathbf{1}_{\{v_i\geq \hat{v}^n\}}}{n}$, then we must have $\gamma^n \geq r^n$. By α-PIR, if $r^n<\alpha$ then $q^n(v)=0$. Thus, we have $P(q^n(v)=1) \leq P(r^n\geq\alpha) \leq P(\gamma^n\geq\alpha)$. Let $P(v_i \geq v_\alpha+\epsilon) := \beta_0$. Since $\{\mathbf{1}_{\{v_i\geq v_\alpha+\epsilon\}}\}_{i=1}^n$ are independent variables and $E(\gamma^n) = E(\frac{\sum_{i=1}^n \mathbf{1}_{\{v_i\geq v_\alpha+\epsilon\}}}{n}) = \beta_0$, we have $P(\gamma^n\geq\alpha) = P(\gamma^n-\beta_0 \geq \alpha-\beta_0) \leq P(|\gamma^n-\beta_0| \geq \alpha-\beta_0) \leq 2exp\{-2n(\alpha-\beta_0)^2\}$, where the last inequality follows from Hoeffding's inequality for sums of bounded random variables. Hence,

$$P(q^n(v) = 1) \leq 2exp\{-2n(\alpha - \beta_0)^2\}. \tag{13}$$

For any given n, the probability of provision $P(q^n(v) = 1)$ is either bounded by inequality (12), or bounded by inequality (13). This implies that $P(q^n(v) = 1) = O(1/n^{\frac{1}{3}})$ for any sequence of mechanisms $\{\{q^n, \{t_i^n\}_{i=1}^n\}\}_{n=1}^{\infty}$ where $\{q^n, \{t_i^n\}_{i=1}^n\} \in \mathcal{M}_\alpha^n$ for each n and $\alpha > 1 - F(c)$. $\square$

We now explain the intuition of the proof of Theorem 2. Fix n and fix an $\epsilon \in (0, c - v_\alpha)$, we have the following two cases: (i) $\hat{v}^n > v_\alpha + \epsilon$, and (ii) $\hat{v}^n \leq v_\alpha + \epsilon$.

In the former case, the break-even valuation $\hat{v}^n$ is relatively high. This implies that the proportion of agents who obtain non-negative interim expected utilities is relatively low. Under the assumption that $\hat{v}^n > v_\alpha + \epsilon$, it can be verified that the above proportion is less than the required agreement rate α with probability one when n is large. Thus, by α-PR, the public good will not be provided in a large economy.

In the latter case, using Lemma 3, Lemma 4 and the fact that $U^n(v_\alpha + \epsilon) \geq U^n(\hat{v}^n) \geq 0$, we can obtain the following upper bound for the expected payment per capita (see also inequality (9) in the proof of Theorem 2):

$$Et_i^n(v_i) \leq (v_\alpha + \epsilon)P(q^n(v) = 1) + C(n) \tag{14}$$

where $C(n)$ approaches zero as n goes to infinity. The above inequality reflects the fact that the *expected payment* that the principal could collect from each agent is not greater than the *base payment* of the agent (which is bounded by $(v_\alpha + \epsilon)P(q^n(v) = 1)$) plus the *expected virtual benefit* of the agent (which is bounded by $C(n)$). The base payment is the tax that the mechanism designer can impose on all agents, and the expected virtual benefit reflects how much additional tax the mechanism designer can extract from a high valuation agent under the information constraint.[17] Since $C(n)$ goes to zero as n goes to infinity, the expected payment that the principal could collect from each agent is at most $(v_\alpha+\epsilon)P(q^n(v)=1)$ in a large economy. However, the assumption that the required agreement rate α is greater than $1-F(c)$ implies that $v_\alpha+\epsilon<c$. Thus, the budget balance constraint cannot be satisfied unless $P(q^n(v)=1)=0$.

The convergence speed of $P(q^n(v)=1)$ toward zero depends on two factors. One is the speed at which the proportion of agents who obtain non-negative interim expected utilities goes to a value that is smaller than α in the first case mentioned above. The other is the speed at which $C(n)$ goes to zero in the second case mentioned above. It turns out that the first speed is on the order of exponential rate by Hoeffding's inequality and the second speed is on the order of $1/n^{\frac{1}{3}}$ as a result of Lemma 4. Thus, $P(q^n(v)=1)$ must converge to zero at a speed not slower than $1/n^{\frac{1}{3}}$.

4 The role of production technologies

In the previous sections, we assumed that the cost of provision is proportional to the number of agents in the economy. In this section, we are going to consider a situation where the per capita cost of provision is decreasing in the number of agents.[18] In particular, we assume that the cost of provision in the n-agent economy is $C(n)=n^\gamma c$, where $\gamma\in[0,1)$ and $c>0$ are constants. The per capita cost of provision is thus $c/n^{1-\gamma}$. In addition, we assume that $\underline{v}=0$.[19]

The purpose of this section is to explore the impact of production technologies on the provision of the public good. As the economy becomes large, there are two *opposite* effects

[17]The fact that $C(n)$ goes to zero as n goes to infinity implies that as the economy becomes large, an agent's expected virtual benefit goes to zero (see also, Hellwig (2003), Norman (2004), etc., for this property)

[18]We will not consider the case in which the per capita cost of provision is increasing in the number of agents, because in that case, as the number of agents increases, the total cost will eventually exceed the maximum possible total benefits of the public good and thus the probability of provision in the first-best mechanism approaches zero. As a result, first-best efficiency can always be achieved.

[19]If $\underline{v}>0$, then since the per capita cost approaches $0<\underline{v}$, first-best efficiency can always be achieved by a mechanism in which the public good is provided with probability 1 and each agent is charged $\underline{v}$.

on the provision of the public good. One is the decreasing per capita cost of provision due to the scale efficiency (measured by $1-\gamma$) in production. The other is the increasing severity of the free-rider problem due to the increase of the number of agents. We will show that, for the case where α is less than 1, no matter how small $1-\gamma$ might be, the positive effect of decreasing cost per capita will eventually outweigh the negative effect of increasing severity of the free-rider problem and first-best efficiency will be achieved. For the case where α equals 1, assuming that v_i is discrete at 0, we find that inefficiency obtains if $1-\gamma$ is less than 1/3.

The analysis for the case where α is less than 1 is straightforward. Observing that $\frac{C(n)}{n}$ approaches zero as n goes to infinity, the threshold $1-F(C(n)/n)$ approaches 1. This implies that the required agreement rate α must be less than the threshold for sufficiently large n. As a result, for any $\gamma \in [0,1)$, first-best efficiency can be achieved in a large economy by mechanisms such as α-referendum.

We now analyze the case where $\alpha = 1$. For simplicity, we assume that the distribution of v_i is continuous on $(0, \overline{v}]$ but is discrete at 0 with $P(v_i = 0) > 0$.[20] For any n, since the required agreement rate equals 1, $\hat{v}^n$ must equal zero. In addition, $v_\alpha = 0$. Thus, we have $\hat{v}^n < v_\alpha + \epsilon = \epsilon$ for any $\epsilon \in (0, c - v_\alpha)$. Letting $\epsilon \to 0$, inequality (9) then implies that for any $0 < \eta < m(v_\alpha + \epsilon) = m(\epsilon) = \min\{P(v_i \leq \epsilon), P(v_i \geq \epsilon)\}$, we have:

$$Et_i^n(v_i) \leq \overline{v}\frac{1}{\sqrt{\eta n}} + 2\bar{v}\eta \tag{15}$$

Notice that η is independent of n. Inequality (15) implies that the expected payment per capita decreases to zero at a speed *not slower* than $1/n^{\frac{1}{3}}$ (this is because the right hand side of inequality (15) is the smallest when $\eta = 4^{-\frac{2}{3}}n^{-\frac{1}{3}}$). Thus, if the cost per capita decreases to zero at a speed *slower* than $1/n^{\frac{1}{3}}$ (i.e., $\gamma \in (2/3, 1)$), then the cost per capita

[20] The assumption that the distribution of v_i has an atom at 0 helps the inefficiency result below (Theorem 3 (ii)). In particular, this assumption ensures that $\frac{v_\alpha + \epsilon}{\sqrt{m(v_\alpha + \epsilon)}} = \frac{\epsilon}{\sqrt{m(\epsilon)}}$ goes to zero as ϵ goes to zero, and thus the upper bound of $Et_i^n(v_i)$ we obtain in the proof of Theorem 2 (in particular, inequality (9)) can be simplified to inequality (15) below.

will eventually exceed the expected payment per capita and the probability of provision must converge to zero. In particular, for $\gamma \in (2/3, 1)$, using the ex ante budget balance constraint $C(n)Eq_i^n(v_i) \leq nEt_i^n(v_i)$ and inequality (15), we have: $Eq_i^n \leq O(1/n^{\gamma-\frac{2}{3}})$, i.e., $P(q^n(v) = 1) \leq O(1/n^{\gamma-\frac{2}{3}})$. Since $1/n^{\gamma-\frac{2}{3}} = \dfrac{1/n^{\frac{1}{3}}}{1/n^{1-\gamma}}$, the convergence rate of the probability of provision (i.e., $1/n^{\gamma-\frac{2}{3}}$) is thus determined by the convergence rate of the expected payment per capita (i.e., $1/n^{\frac{1}{3}}$) and the convergence rate of the cost per capita (i.e., $1/n^{1-\gamma}$).

Finally, if $\alpha = 1$ and $\gamma \in [0, 1/2)$, then first-best efficiency can always be achieved in a large economy (see, Hellwig (2003)).[21]

In summary, we have:

Theorem 3. *Assume the cost of provision in the n-agent economy is $n^\gamma c$, then,*

(i) if $\alpha \in [0, 1)$ and $\gamma \in [0, 1)$ or $\alpha = 1$ and $\gamma \in [0, 1/2)$, the first-best provision level can be achieved asymptotically by a sequence of mechanisms that satisfy INTIC, EXABB, and α-PIR;

(ii) if $\alpha = 1$ and $\gamma \in (2/3, 1)$, assuming that v_i is continuous on $(0, \overline{v}]$ and is discrete at 0 (i.e., v_i has an atom at 0), the probabilities of provision in any sequence of anonymous mechanisms that satisfy INTIC, EXABB, and α-PIR approach zero as $n \to \infty$ at a speed not slower than $1/n^{\gamma-\frac{2}{3}}$.

The case where $\alpha = 1$ and $\gamma \in [1/2, 2/3]$ is not covered in Theorem 3. In this case, we can neither prove nor disprove whether efficiency or inefficiency obtains. Our conjecture is that whether efficiency obtains depends on the distributions of the valuations of the agents.

[21] Actually, Hellwig (2003) assumed that the total cost of provision is independent of the number of agents (for the case where the first-best provision level is bounded). He constructed a sequence of mechanisms and showed that first-best efficiency can be achieved by this sequence of mechanisms (Proposition 3, Hellwig 2003). It can be easily verified that the sequence of mechanisms constructed by Hellwig (2003) also attains first-best efficiency as soon as the cost per capita converges to zero at a speed that is faster than $1/\sqrt{n}$.

5 Conclusions

This paper considers the public good provision problem in which the public good can be provided and payments can be collected from agents only if the proportion of agents who obtain nonnegative interim expected utilities from the mechanism weakly exceeds a required agreement rate α. We show that, if α is less than $1 - F(c)$, then there exists a sequence of α-referenda satisfying EXPIC, EXPBB and α-PIR such that as the size n of the economy becomes large, the probabilities of the public good being provided in α-referenda approach 1 at a speed not slower than the exponential rate; and if α exceeds $1-F(c)$, then the probabilities that the public good will be provided in any sequence of mechanisms that satisfy INTIC, EXABB, and α-PIR approach 0 at a speed not slower than $1/n^{\frac{1}{3}}$. Hence, this paper not only obtains asymptotic efficiency/inefficiency results for various required agreement rates settings, but also characterizes the speed at which the probability of provision reaches its efficient/inefficient level. In addition, as a methodological contribution, we propose the standard deviation of an agent's interim expected provision as a measure of the agent's influence in a mechanism. We characterize the convergence speed of an agent's influence in any sequence of anonymous mechanisms toward zero (Lemma 2) and the convergence speed of any sequence of anonymous feasible mechanisms toward a constant mechanism (Lemma 3 and Lemma 4).

Our results imply that $F(c)$ is the minimum fraction of the agents that need to be coerced into participation if we want to obtain efficiency asymptotically. An interesting question to ask is whether we can design a mechanism to elicit payments and also determine which agents need to be coerced in order to achieve efficiency. We do not have a clear answer to this question in the finite economy, but we do have an answer when the economy is large. In particular, when the economy becomes large, our Lemmas 3 and 4 show that any mechanism satisfying INTIC, EXABB and α-PIR must converge to a mechanism where the interim expected payment and provision functions are constant. By budget balance, this implies that if the public good is going to be provided, then the interim expected payment of an agent in any mechanism must be roughly equal to the per capita cost c, independent of the agent's valuation. This implies that if we want to obtain efficiency, then (roughly speaking), any agent with valuation less than c should be coerced into participation no matter what mechanism we are going to use.

Appendix 1

Proof of Theorem 1:

(i) It can be easily verified that the mechanism $M_\alpha^n = \{\tilde{q}^{n,\alpha}, \tilde{t}_i^{n,\alpha}\}$ satisfies EXPIC, EXPBB, and α-PIR (using the fact that $\tilde{U}_i^{n,\alpha}(c) = 0$).

(ii) Let $\beta = 1 - F(c)$.

Note $\{\mathbf{1}_{\{v_i \geq c\}}\}_{i=1}^n$ are independent variable and $E(\frac{\sum_{i=1}^n \mathbf{1}_{\{v_i \geq c\}}}{n}) = \beta$, then we have:

$$\begin{aligned} P(\tilde{q}^{n,\alpha} = 1) = P(\frac{\sum_{i=1}^n \mathbf{1}_{\{v_i \geq c\}}}{n} \geq \alpha) &\geq P(\left|\beta - \frac{\sum_{i=1}^n \mathbf{1}_{\{v_i \geq c\}}}{n}\right| \leq \beta - \alpha) \\ &\geq 1 - P(\left|\beta - \frac{\sum_{i=1}^n \mathbf{1}_{\{v_i \geq c\}}}{n}\right| \geq \beta - \alpha) \\ &\geq 1 - 2exp\{-2n(\beta - \alpha)^2\} \qquad (16) \end{aligned}$$

where the last inequality follows from Hoeffding's inequality, which says that $P(\left|\bar{X} - E(\bar{X})\right| \geq t) \leq 2exp(-2nt^2)$ where $t > 0$ is a constant and $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ is the average of a sequence of i.i.d. random variables $\{X_i\}_{i=1}^n$ with $Pr(X_i \in [0,1]) = 1$ (see, e.g., Hoeffding (1963)).

Thus, we have $1 - P(\tilde{q}^{n,\alpha} = 1) \leq 2exp\{-2n(\beta - \alpha)^2\}$.

Proof of Lemma 2:[22]

[22] The technique used in the proof also appeared in Becker (2012). In particular, inequality (18) below may be regarded as a special case of Proposition 2.1 in Becker (2012).

Let $V = [\underline{v}, \overline{v}]^n$ and $V_i = [\underline{v}, \overline{v}]$ for any i. We have:

$$
\begin{aligned}
& \int_V [q^n(v) - Eq^n(v) - \sum_i (q_i^n(v_i) - Eq_i^n(v_i))]^2 dF(v_1) \cdots F(v_n) \\
= & \int_V [q^n(v) - Eq^n(v)]^2 dF(v_1) \cdots F(v_n) \\
& -2 \sum_i \int_V (q^n(v) - Eq^n(v))(q_i^n(v_i) - Eq_i^n(v_i)) dF(v_1) \cdots F(v_n) \\
& + \sum_i \int_V [q_i^n(v_i) - Eq_i^n(v_i)]^2 dF(v_1) \cdots F(v_n) \\
= & \int_V [q^n(v) - Eq^n(v)]^2 dF(v_1) \cdots F(v_n) - 2 \sum_i \int_{V_i} [q_i^n(v_i) - Eq_i^n(v_i)]^2 dF(v_i) \\
+ & \sum_i \int_{V_i} [q_i^n(v_i) - Eq_i^n(v_i)]^2 dF(v_i) \\
= & Var(q^n(v)) - \sum_i Var(q_i^n(v_i)) \\
= & Var(q^n(v)) - nVar(q_i^n(v_i))
\end{aligned}
\tag{17}
$$

where the first equality follows from the fact that $q_1^n(v_1), \cdots, q_n^n(v_n)$ are independent, and the second equality follows from the fact that $Eq^n(v) = Eq_i^n(v_i)$ for any i.

Using equality (17) and the fact that $\int_V [q^n(v) - Eq^n(v) - \sum_i (q_i^n(v_i) - Eq_i^n(v_i))]^2 dF(v_1) \cdots F(v_n) \geq 0$, we have:

$$
Var(q_i^n(v_i)) \leq \frac{1}{n} Var(q^n(v)). \tag{18}
$$

Notice that $0 \leq q^n(v) \leq 1$ for any n and $v \in V$, so $Var(q^n(v)) \leq 1$. Thus, $Var(q_i^n(v_i)) = O(1/n)$.

Proof of Lemma 3:

First, by Lemma 1, if $\{q^n, \{t_i^n\}_{i=1}^n\}$ satisfies INTIC, then $q_i^n(v_i)$ is nondecreasing on $[\underline{v}, \overline{v}]$.

For a given $\varepsilon > 0$, using Chebyshev's inequality, Lemma 2 and the fact that $Var(q^n(v)) \leq$

$\frac{1}{4}$ (note $q^n(v)$ follows a Bernoulli distribution that can take two values: 0 and 1), we have:

$$P(|q_i^n(v_i) - Eq_i^n(v_i)| \geq \varepsilon) \leq \frac{Var(q_i^n(v_i))}{\varepsilon^2} \leq \frac{1}{4n\varepsilon^2}.$$

Fix $\tilde{v} \in (\underline{v}, \bar{v})$. If $n \geq \frac{1}{4m(\tilde{v})\varepsilon^2}$, then

$$P(|q_i^n(v_i) - Eq_i^n(v_i)| > \varepsilon + \varepsilon_0) \leq \frac{1}{4n(\varepsilon + \varepsilon_0)^2} < m(\tilde{v}) \quad \text{for any } \varepsilon_0 > 0. \tag{19}$$

Now if $|q_i^n(\tilde{v}) - Eq_i^n(v_i)| > \varepsilon + \varepsilon_0$, then we have either $q_i^n(\tilde{v}) - Eq_i^n(v_i) > \varepsilon + \varepsilon_0$ or $q_i^n(\tilde{v}) - Eq_i^n(v_i) < -(\varepsilon + \varepsilon_0)$. Since $q_i^n(v_i)$ is a nondecreasing function, we have either $q_i^n(v_i) - Eq_i^n(v_i) > (\varepsilon + \varepsilon_0)$ for all $v_i \geq \tilde{v}$ or $q_i^n(v_i) - Eq_i^n(v_i) < -(\varepsilon + \varepsilon_0)$ for all $v_i \leq \tilde{v}$. Thus, $P(|q_i^n(v_i) - Eq_i^n(v_i)| > \varepsilon + \varepsilon_0) \geq \min(P(v_i \geq \tilde{v}), P(v_i \leq \tilde{v})) = m(\tilde{v})$ which is a contradiction with (19). As a result, we must have $|q_i^n(\tilde{v}) - Eq_i^n(v_i)| \leq \varepsilon + \varepsilon_0$ for all $n \geq \frac{1}{4m(\tilde{v})\varepsilon^2}$. Since $\varepsilon_0 > 0$ is arbitrary, then $|q_i^n(\tilde{v}) - Eq_i^n(v_i)| \leq \varepsilon$ for all $n \geq \frac{1}{4m(\tilde{v})\varepsilon^2}$. Hence, for any given N, we have $|q_i^n(\tilde{v}) - Eq_i^n(v_i)| \leq \frac{1}{2\sqrt{m(\tilde{v})N}}$ for all $n \geq N$. This implies that $|q_i^n(\tilde{v}) - Eq_i^n(v_i)| \leq \frac{1}{2\sqrt{m(\tilde{v})n}}$ for any given n.

Proof of Lemma 4:

We first show that there is some $v_i \in [\underline{v}, \bar{v}]$ such that $U^n(v_i) \geq 0$. Suppose this is not true, then we must have $U^n(v_i) < 0$ for all $v_i \in [\underline{v}, \bar{v}]$. This implies that $r^n(v) = 0$ for any $v \in [\underline{v}, \bar{v}]^n$. Since $r^n(v) = 0 < \alpha$, we must have $q^n(v) = 0$ and $t_i^n(v) = 0$ for all i and all $v \in [\underline{v}, \bar{v}]^n$ by α-PIR. This implies that $U^n(v_i) = 0$ for all $v_i \in [\underline{v}, \bar{v}]$. Contradiction!

We thus have shown that there is some $v_i \in [\underline{v}, \bar{v}]$ such that $U^n(v_i) \geq 0$. Now, by the definition of $\hat{v}^n$, we must have $\hat{v}^n \in [\underline{v}, \bar{v}]$ and $U^n(\hat{v}^n) \geq 0$.

We next show that $t_i^n(v_i)$ is bounded. On one hand, we have $t_i^n(v_i) = v_i q_i^n(v_i) - \int_{\hat{v}^n}^{v_i} q_i^n(s)ds - U^n(\hat{v}^n) \leq \bar{v}$, where the equality follows from the fact that $\{q^n, \{t_i^n\}_{i=1}^n\}$ satisfies INTIC and the inequality follows from the facts that $U^n(\hat{v}^n) \geq 0$ and $q_i^n(v_i) \geq 0$. On the other hand, by EXABB, we have $Et_i^n(v_i) \geq cEq_i^n(v_i)$. This implies that $E[v_i q_i^n(v_i)] - E[\int_{\hat{v}^n}^{v_i} q_i^n(s)ds] - U^n(\hat{v}^n) \geq cEq_i^n(v_i)$, i.e., $U^n(\hat{v}^n) \leq E[v_i q_i^n(v_i)] - E[\int_{\hat{v}^n}^{v_i} q_i^n(s)ds] - cEq_i^n(v_i) \leq \bar{v}$. This latter inequality and the fact that $\int_{\hat{v}^n}^{v_i} q_i^n(s)ds \leq \int_{\hat{v}^n}^{v_i} q_i^n(v_i)ds$ (because q_i^n is a non-decreasing function by INTIC) imply that $t_i^n(v_i) = v_i q_i^n(v_i) - \int_{\hat{v}^n}^{v_i} q_i^n(s)ds - U^n(\hat{v}^n) \geq v_i q_i^n(v_i) - \int_{\hat{v}^n}^{v_i} q_i^n(v_i)ds - U^n(\hat{v}^n) = v_i q_i^n(v_i) - (v_i - \hat{v}^n) q_i^n(v_i) - U^n(\hat{v}^n) = \hat{v}^n q_i^n(v_i) - U^n(\hat{v}^n) \geq -\bar{v}$ (the last inequality is because $\hat{v}^n \in [\underline{v}, \bar{v}] \subset R_+$). We thus have shown that $|t_i^n(v_i)| \leq \bar{v}$ for any $v_i \in [\underline{v}, \bar{v}]$.

Next, we prove the lemma. Let $\bar{\epsilon} > 0$ be such that $P(v_i \leq \underline{v} + \bar{\epsilon}) = P(v_i \geq \underline{v} + \bar{\epsilon})$. For any $\epsilon \in (0, \bar{\epsilon})$, define $\mu(\epsilon)$ as the unique value such that $P(v_i \geq \bar{v} - \mu(\epsilon)) = P(v_i \leq \underline{v} + \epsilon)$. Define $h(\epsilon) = P(v_i \leq \underline{v} + \epsilon)$. Notice that as $\epsilon \to 0$, we have $\mu(\epsilon) \to 0$ and $h(\epsilon) \to 0$. Fix $\tilde{v} \in (\underline{v} + \epsilon, \bar{v} - \mu(\epsilon))$. For any $v_i \in (\underline{v} + \epsilon, \bar{v} - \mu(\epsilon))$, we have $t_i^n(v_i) \leq v_i(q_i^n(v_i) - q_i^n(\tilde{v})) + t_i^n(\tilde{v}) \leq \bar{v}\frac{1}{\sqrt{h(\epsilon)n}} + t_i^n(\tilde{v})$, where the first inequality follows from the incentive compatibility constraint and the second inequality follows from Lemma 3. Thus, $Et_i^n(v_i) = \int_{(\underline{v}+\epsilon, \bar{v}-\mu(\epsilon))} t_i^n(v_i)dF_i(v_i) + \int_{[\underline{v}, \underline{v}+\epsilon]} t_i^n(v_i)dF_i(v_i) + \int_{[\bar{v}-\mu(\epsilon), \bar{v}]} t_i^n(v_i)dF_i(v_i) \leq \bar{v}\frac{1}{\sqrt{h(\epsilon)n}} + t_i^n(\tilde{v}) + 2\bar{v}h(\epsilon)$. That is, $t_i^n(\tilde{v}) - Et_i^n(v_i) \geq -\bar{v}\frac{1}{\sqrt{h(\epsilon)n}} - 2\bar{v}h(\epsilon)$. Similarly, it can be shown that $t_i^n(\tilde{v}) - Et_i^n(v_i) \leq \bar{v}\frac{1}{\sqrt{h(\epsilon)n}} + 2\bar{v}h(\epsilon)$. Thus, $|t_i^n(\tilde{v}) - Et_i^n(v_i)| \leq \bar{v}\frac{1}{\sqrt{h(\epsilon)n}} + 2\bar{v}h(\epsilon)$ for any $\tilde{v} \in (\underline{v} + \epsilon, \bar{v} - \mu(\epsilon))$. This implies that for any $\tilde{v} \in (\underline{v}, \bar{v})$ and for any $0 < \eta < m(\tilde{v}) = \min\{P(v_i \leq \tilde{v}), P(v_i \geq \tilde{v})\}$, we have $|t_i^n(\tilde{v}) - Et_i^n(v_i)| \leq \bar{v}\frac{1}{\sqrt{\eta n}} + 2\bar{v}\eta$.

Appendix 2

On convergence rates of expected provision level and per capita expected payment toward zero in Al-Najjar and Smorodinsky (2000):

In Al-Najjar and Smorodinsky (2000), the public good provision mechanism is denoted by (δ, c), where $c = (c_1, ..., c_N)$ and N is the number of agents. Note that $\delta : \Omega \to \{0, 1\}$ is the provision function and $c_i : \Omega \to R$ is the payment function for agent i. Ω is the underlying type space of all agents.

Al-Najjar and Smorodinsky (2000) assume that the distribution of any agent's valuation is *continuous* on $(0, \bar{t}]$, and is *discrete* at 0 with the probability at 0 being greater than ϵ for some $\epsilon > 0$ for all agents.[23] The proposition on page 331 in Al-Najjar and Smorodinsky (2000) says that $\sup_{\delta,c} E\delta \leq \frac{t^+}{\beta}[R_{\eta,N} + \eta]$ for *any* $0 < \eta \leq \epsilon$, where $R_{\eta,N} = \frac{1}{\sqrt{\eta\pi}}\frac{1}{\sqrt{N}}$ and β is a constant. For simplicity, we drop the constraint $\eta \leq \epsilon$. It should be noted that imposing the constraint $\eta \leq \epsilon$ will not change the result below. Now, define:

$$\eta^*(N) = \underset{\eta>0}{argmin}\, \frac{t^+}{\beta}[R_{\eta,N} + \eta]$$

Then, the *best* bound for $\sup_{\delta,c} E\delta$ is $\frac{t^+}{\beta}[R_{\eta^*(N),N} + \eta^*(N)]$. The first-order condition of the above minimization problem can be written as $-\frac{1}{2}\eta^{-\frac{3}{2}}\frac{1}{\sqrt{\pi}\sqrt{N}} + 1 = 0$ (it can be verified that the second-order condition for the problem is satisfied). That is, $\eta^*(N) = \frac{1}{(4\pi N)^{\frac{1}{3}}}$. Thus, we have $\sup_{\delta,c} E\delta \leq \frac{t^+}{\beta}[R_{\eta^*(N),N} + \eta^*(N)] = \frac{t^+}{\beta}(4^{\frac{1}{6}} + 4^{-\frac{1}{2}})\pi^{-\frac{1}{3}}\frac{1}{N^{\frac{1}{3}}} = O(N^{-\frac{1}{3}})$.

Thus, the convergence rate of $\sup_{\delta,c} E\delta$ toward zero is on the order of $N^{-\frac{1}{3}}$, not $N^{-\frac{1}{2}}$ as stated on page 331 of Al-Najjar and Smorodinsky (2000). That is, the probability of the public good being provided converges to zero at a rate of $N^{-\frac{1}{3}}$.

Now for the bound on expected payments, from page 342 of Al-Najjar and Smorodinsky (2000), we have that $E(c_n) \leq t^+[V_n(\delta, \eta) + \eta]$ for any $0 < \eta \leq \epsilon$. Thus, $\frac{\sum Ec_n}{N} \leq t^+[R_{\eta,N} + \eta] \leq t^+[\frac{1}{\sqrt{\eta\pi}}\frac{1}{\sqrt{N}} + \eta]$ and $\sum Ec_n \leq t^+[\frac{1}{\sqrt{\eta\pi}}\sqrt{N} + \eta N]$ for any $0 < \eta \leq \epsilon$. By a similar proof, we have $\frac{\sum Ec_n}{N} \leq O(N^{-\frac{1}{3}})$. That is, the per capita expected payment converges to zero at a speed not slower than $N^{-\frac{1}{3}}$. As a result, the expected aggregated payment is $O(N^{\frac{2}{3}})$.

[23] Al-Najjar and Smorodinsky (2000) also considered the case where the distribution of agents' valuations is fully discrete and in that case, the convergence rate of expected provision toward zero is indeed $N^{-\frac{1}{2}}$ (see Corollary in Al-Najjar and Smorodinsky (2000)).

References

N. I. Al-Najjar, R. Smorodinsky, *Pivotal players and the characterization of influence*, Journal of Economic Theory 92 (2000), 318-342.

J. G. Becker, *A note on the number of α-pivotal players*, Mathematics of Operations Research 37 (2012), 196-200.

T. Börgers, P. Norman, *A note on budget balance under interim participation constraints: the case of independent types*, Economic Theory 39 (2009), 477-489.

E. H. Clarke, *Multipart pricing of public goods*, Public Choice II (1971), 19-33.

C. d'Aspremont, L. A. Gérard-Varet, *Incentives and incomplete information*, Journal of Public Economics 11 (1979), 25-45.

T. Groves, *Incentives in teams*, Econometrica 14 (1973), 617-631.

M. F. Hellwig, *Public-good provision with many participants*, Review of Economic Studies 70 (2003), 589-614.

W. Hoeffding, *Inequalities for sums of bounded random variables*, Journal of the American Statistical Association 58 (1963), 13-30.

J. O. Ledyard, T. R. Palfrey, *The approximation of efficient public good mechanisms by simple voting schemes*, Journal of Public Economics 83 (2002), 153-171.

G. Mailath, A. Postlewaite, *Asymmetric information bargaining problems with many agents*, Review of Economic Studies 57 (1990), 531-567.

R. B. Myerson, *Optimal auction design*, Mathematics of Operations Research 6 (1981), 58-73.

P. Norman, *Efficient mechanisms for public goods with use exclusions*, Review of Economic Studies 71 (2004), 1163-1188.

R. Rob, *Pollution claim settlements under private information*, Journal of Economic Theory 47 (1989), 307-333.