Public Good Provision with Constitutional Constraint

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Abstract: This paper studies the public good provision problem, in which the principal faces a constitutional constraint in the sense that in order for a public good provision mechanism to be implemented, it must first be approved by agents under a prespecified voting rule. We find that as long as the voting rule is not the unanimity rule, the principal can propose a mechanism such that first-best efficiency of provision of the public good is achieved. We also consider various constraints, such as prohibition of discriminatory mechanisms or the existence of vote buying, and discuss optimal voting rules in these situations.

Keywords: Public goods; Constitutional constraint; Voting; Efficiency.

JEL classification: D82, H41, D72

1. Introduction

This paper considers the public good provision problem. Imagine that a mayor of a city (or a principal) is planning to provide a public good (e.g., build a public library) in the city. There is a large group of citizens (or agents) in the city. Each agent has a valuation about the public good, which is unknown to the principal. The principal can *propose* a mechanism, which asks each agent to report his valuation, and based on all agents' reported

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valuations, the mechanism decides whether to provide the public good and how to distribute the cost of the public good among the agents if the public good is provided. We assume that the principal faces a constitutional constraint in the sense that the *implementation* of any mechanism proposed by the principal requires the approval of the agents under a pre-specified α-majority voting rule. This reflects the fact that in reality, implementation of a new law in a city usually requires the approval of the city council (which can be regarded as a committee that represents the agents in the city). Our main research question is: For any given constitutional constraint (i.e., for any given voting rule), can the principal propose a suitably designed mechanism such that first-best efficiency of provision of the public good is achieved?

Our main model is a private value model, in which agents' valuations are i.i.d. The public good is a discrete good, which can either be provided or not provided. We assume that the expected value of an agent's valuation exceeds the average cost of the public good, which implies that it is common knowledge that the public good *should* be provided in a large economy.

Suppose that the voting rule is the α-majority rule (i.e., α fraction of yes votes is required to get a mechanism approved). We find that as long as α is less than one, then the principal can always design a mechanism that will be approved by agents with probability one in a large economy. In addition, conditional on approval of the mechanism, the probability that the public good will be provided goes to one as the economy becomes large. We thus obtain an efficiency result for any given $\alpha < 1$.

We next briefly explain how we obtain the efficiency result and describe the difficulties. To achieve efficiency, a prerequisite is that the mechanism proposed by the principal can be supported by more than α fraction of agents. Note that an agent votes for a mechanism if and only if voting for the mechanism yields a higher payoff than voting against the mechanism, *conditional on the event that the agent's vote is pivotal*. So, in order to induce an agent to vote for a mechanism, we can simply reward the agent by giving him a sufficiently large cash payment if he votes yes *and* his vote is pivotal. This is a key idea that we will use to construct the mechanism that achieves efficiency. However, certain difficulties arise.

First, it appears that as long as the reward given to a pivotal agent who votes yes is sufficiently large, all agents will choose to vote for the mechanism. However, if all agents vote for the mechanism, then no agent is pivotal (except for the extreme case in which the voting rule is the unanimity rule). As a result, all agents voting for the mechanism is a *weak equilibrium* in the sense that every agent is indifferent between voting for the mechanism and voting against it. Such an equilibrium may require a high level of coordination and is quite fragile. In this paper, we will exclude such weak equilibrium. To obtain a *strict equilibrium*, we will need to have some (maybe a very small fraction of) agents vote against the mechanism so that the event that an agent is pivotal occurs with a non-zero probability.

Second, agents' voting behavior is usually anonymous. So, the principal cannot reward an agent directly based on the agent's voting behavior. However, the principal can indirectly reward an agent based on the agent's reported valuation by using discriminatory payments when the mechanism is approved. Thus, the mechanism proposed by the principal should be such that in equilibrium, an agent's voting behavior and his reported valuation are "correlated" so that the principal can (indirectly) reward the agent who votes for the mechanism.

We find a very simple mechanism, called the *Equal Cost Sharing with Pivotal Reward mechanism* (or simply the ECSPR mechanism), that addresses the issues mentioned above. The mechanism is as follows. Let n^* be the minimum number of agents required for a mechanism to be approved (i.e., n^* is the smallest integer that is greater than or equal to αn, where n is the number of agents in the economy). The ECSPR mechanism, once approved, asks each agent to report either a high value h or a low value l.[2] If there are exactly n^* agents who report high values, then the public good is provided with some probability, and any agent who reports the high value will be "rewarded" with paying less than the

[2]Obviously, the mechanism is an indirect mechanism. We can transform it to a truthful direct mechanism using the revelation principle. However, it is more convenient to state the mechanism as an indirect mechanism.

average cost if the public good is provided (where the difference between the average cost and the payment made by an agent who reports the high value in this case is called the *pivotal reward*).[3] If more than n^* agents report high values, the public good is provided with probability one and its cost is equally distributed among all agents. If exactly $n^* - 1$ agents report high values, the public good is not provided, but the agents who report low values need to make some payments to the agents who report high values. In all other cases, the public good is not provided and no payment is collected from any agent.

We find that if the principal proposes the ECSPR mechanism (with suitably chosen pivotal reward), then there is a strict Bayesian Nash equilibrium in the voting game. In this equilibrium, any agent whose valuation exceeds a threshold will vote for the mechanism and report h, and any agent whose valuation is less than the threshold will vote against the mechanism and report l. We find that as the pivotal reward increases, the threshold decreases. As a result, there exists a range of pivotal rewards such that for any pivotal reward in this range, (i) the threshold is low enough such that the probability that an agent's valuation exceeds the threshold is greater than α, and (ii) the threshold exceeds the lowest possible valuation of an agent such that some very low-valuation agents still prefer to vote against the mechanism. Interestingly, this range is independent of the number of agents in the economy. We can use the law of large numbers to show that, as the economy becomes large, as long as the pivotal reward remains within the above range, the mechanism will be approved with probability one and the provision of the public good will approach its first-best efficient level.

1.1. Literature Review

A central topic in the public good provision literature concerns whether public goods can be provided at an efficient level. Various positive and negative results have emerged. Mailath and Postlewaite (1990) find that a public good will never be provided in a large

[3] Correspondingly, any agent who reports a low value in this case will make a payment that is greater than the average cost of the public good.

economy if the individual rationality (IR) constraint is required (see also Rob 1989). In the case in which the IR constraint is not a concern, d'Aspremont and Gérard-Varet (1979) show that a public good can be provided efficiently under the so-called AGV mechanism.[4] Hellwig (2003) obtains an efficiency result by considering the situation in which the IR constraint is required, but the cost of the public good is fixed and independent of the number of agents. Norman (2004) studies a public good provision problem with use exclusions, and obtains both efficiency and inefficiency results, depending on whether the average cost of the public good is less than or exceeds a threshold.

This paper is also related to Rong (2014), who proposes a so-called α individual rationality constraint, in which the IR constraint is only required for α fraction of agents. Rong (2014) shows that the efficiency result is obtained if α is less than a threshold, while the inefficiency result is obtained if α exceeds the threshold. A key difference between Rong (2014) and this paper is that in the former, a mechanism can be implemented if and only if α fraction of agents obtain non-negative expected payoffs from the mechanism, while in this paper, a mechanism can be implemented if and only if α fraction of agents vote for the mechanism. Note that the fact that an agent votes for a mechanism does not necessarily mean that the agent will obtain a non-negative expected payoff from the mechanism; it only means that *in the pivotal event* (i.e., in the case where the agent's vote is pivotal), the implementation of the mechanism yields a non-negative expected payoff for the agent.[5] In other words, the fact that an agent's IR constraint is satisfied does not necessarily mean that the agent will vote for the mechanism (likewise, if an agent chooses to vote for a mechanism, it does not necessarily mean that the agent's IR constraint is satisfied).

Finally, this paper is closely related to Dal Bó (2007), who studies the problem of how

[4] The problems studied in these papers may be regarded as extreme cases of our model. In particular, the problems studied by Mailath and Postlewaite (1990) and Rob (1989) roughly correspond to the case of $\alpha = 1$ in our model (see Corollary 1 (i) for a more detailed analysis), while the problem studied by d'Aspremont and Gérard-Varet (1979) corresponds to the case of $\alpha = 0$ in our model (although the AGV mechanism is different from the ECSPR mechanism proposed in this paper).

[5] In addition, whether an agent's vote is pivotal depends on the strategies used by other agents and their valuations.

an outside party can influence a committee's voting decision on an issue (e.g., a project), which is assumed to be *undesirable* for all members of the committee. Dal Bó (2007) finds that if the outside party can reward the decisive votes differently, then it can influence the committee's voting decision by promising each committee member a sufficiently large reward in the case in which the committee member's vote is pivotal and the member votes yes (in equilibrium, all committee members will vote yes, so there is no cost for making such a promise). Such rewards in the pivotal event are similar to the pivotal reward proposed in this paper. However, a major difference between Dal Bó's model and ours is that in his, the reward to a committee member is directly based on the member's voting decision. This practice is usually illegal, and thus not feasible in reality. In contrast, the reward to an agent in our model is based on the agent's reported valuation.

The paper is organized as follows. Section 2 considers the basic model, and Section 3 considers an extension in which there is a common value component in agents' valuations. Section 4 discusses the choice of voting rule in situations in which the ECSPR mechanism does not work properly. Section 5 concludes.

2. The Basic Model

2.1. Setup

Suppose that a principal wants to build a public good (e.g., a public library) in a community, in which there are n agents. The public good is a discrete commodity that will either be provided or not provided. The cost of providing the public good is $C(n)$. We assume that $C(n)$ is (weakly) increasing in n. This assumption also reflects the fact that to serve a larger community, the principal needs to construct a larger library, which incurs a greater cost. Given that n is fixed for most parts of the paper, we write $C(n)$ as C when there is no confusion. We denote agent i's valuation about the public good by v_i, which is known only to agent i. $v_1, \cdots, v_n$ are independent and identically distributed in accordance with a common distribution function F, which is common knowledge among all agents and the principal. The support of F is denoted by $[\underline{v}, \overline{v}]$. We assume that the density function of F is continuous and strictly positive on $[\underline{v}, \overline{v}]$. To make the problem nontrivial, we assume that the average cost of the public good exceeds the lowest possible valuation of an agent, i.e., $\frac{C(n)}{n} > \underline{v}$. Finally, we assume that $\frac{C(n)}{n} < v^e$, where $v^e = Ev_i$. This assumption implies that the public good *should* be provided in a large economy, because $\frac{\sum v_i}{n} \to v^e$ as $n \to \infty$ by the law of large numbers. Since we mainly focus on a large economy,[6] first-best efficiency means that the public good should always be provided (in Section 3, we will consider an extended setup in with there is a common value component in agents' valuations and thus both provision and non-provision of the public good may be optimal).

The principal can propose a public good provision mechanism. A *mechanism* is a function pair $\{q, \{t_i\}_{i=1}^n\}$, where $q : [\underline{v}, \overline{v}]^n \to [0, 1]$ is the probability that the public good is provided and $t_i : [\underline{v}, \overline{v}]^n \to R$ is the payment collected from agent i.[7] Note that q and t_i are functions of the reported valuations of all agents. Agent i obtains an (ex post) utility of $v_i q - t_i$ if the mechanism $\{q, \{t_i\}_{i=1}^n\}$ is implemented.

We do not directly impose any individual rationality constraint on the mechanism. Instead, we assume that the principal faces a *constitutional constraint*, in the sense that in order for a mechanism to be implemented, the mechanism must first be approved by agents under a prespecified α-majority voting rule. To make the problem nontrivial, we assume that $\alpha > 1 - F(\frac{C}{n})$ (if $\alpha < 1 - F(\frac{C}{n})$, then the principal can simply propose the equal cost sharing mechanism, which will be approved under the α-majority voting rule because $1 - F(\frac{C}{n})$ fraction of agents will support the mechanism in a large economy).

We impose two additional requirements on the mechanism $\{q, \{t_i\}_{i=1}^n\}$. First, we require that the mechanism be *anonymous*. That is, q and t_i do not depend on the identities of agents. This assumption is needed to avoid the uninteresting case in which the principal

[6]In the public good provision environment, it is natural to assume that n is large.

[7]We allow both indirect mechanisms and direct mechanisms. However, for the sake of expositional clarity, we use direct mechanisms when describing mechanisms in general (also, note that according to the revelation principle, any indirect mechanism can be equivalently represented as a truthful direct mechanism).

simply asks a specific agent to pay all the costs of the public good, and such a mechanism will be approved by agents under any α-majority rule as long as $\alpha \leq (n-1)/n$. Second, we require that the mechanism satisfy the *ex post budget balance* constraint.[8] That is, $\sum_{i=1}^{n} t_i(\hat{v}_1, ..., \hat{v}_n) - Cq(\hat{v}_1, ..., \hat{v}_n) = 0$ for any reported valuations $(\hat{v}_1, ..., \hat{v}_n)$.

The game considered in this paper is called the *voting game*. Its timing is as follows.

1. The principal proposes a public good provision mechanism $\{q, \{t_i\}_{i=1}^{n}\}$.
2. Each agent knows his own valuation about the public good, without knowing other agents' valuations. Each agent chooses whether to vote for the mechanism or vote against it, and reports his valuation. If at least α fraction of agents vote for the mechanism, the game proceeds to the next step. Otherwise, the game ends immediately (in this case, the public good is not provided and there is no payment collected from any agent).[9]
3. The mechanism is implemented based on the reported valuations of all agents.

We can also modify the above game such that agents report their valuations *after* approval of the mechanism, and this modification will not change agents' equilibrium behavior. This is because in the game in which an agent makes his voting decision and reports his valuation at the same time, the agent knows that his reported valuation will be used only after the mechanism is approved.

Notice that our definition of the mechanism $\{q, \{t_i\}_{i=1}^{n}\}$ requires that the mechanism only depends on the agents' reported valuations and does not depend on the agents' voting behavior. This assumption is natural, because voting is usually anonymous and thus the

[8]Note that ex post budget balance (EXPBB) is a stronger requirement than ex ante budget balance (EXABB). However, it is also well known in the literature that in Bayesian mechanism design, if there are two or more agents and agents' types are independent, then EXABB and EXPBB are equivalent in the sense that for any mechanism that satisfies EXABB, we can always construct side payments to ensure that EXPBB is satisfied and that any agent obtains the same interim expected payoff as before (see, e.g., Börgers and Norman 2009).

[9]Grüner and Koriyama (2012) consider a model where if a public good provision mechanism is vetoed, then the situation of a majority voting with equal cost sharing will be enforced. In contrast, our model assumes that if a mechanism is vetoed, there is no "renegotiation" and all agents obtain a payoff of zero.

principal does not know which agent votes yes and which agent votes no (even when voting is not anonymous, the principal is usually prohibited from discriminating between agents based on their voting behavior).

Our main research question is, for any given α-majority rule, can the principal propose a mechanism that will be approved by agents with a probability close to one and (conditional on approval of the mechanism) the public good will be provided with a probability close to one?

2.2. Equilibrium notion

As is typical in any game that involves voting, our voting game may have some uninteresting equilibria. For example, assuming that $\alpha > 1/n$, for *any* given mechanism proposed by the principal, a trivial equilibrium is that all agents vote against the mechanism (regardless of their valuations), and thus the public good will never be provided.[10] Such an equilibrium is uninteresting, because it requires a very high level of coordination between agents. More precisely, in such an equilibrium, any agent is actually indifferent between voting for the mechanism and voting against it. To avoid such an equilibrium, the equilibrium concept we will use is the *strict Bayesian Nash equilibrium* (or simply the *strict BNE*). The strict BNE is a BNE with the additional requirement that for almost any type of any agent, if the agent deviates from his equilibrium action to any other action, the agent will be strictly worse off (rather than "weakly worse off," as in the standard BNE). More precisely, agents' strategy profile $(\sigma_1, ..., \sigma_n)$ forms a strict BNE if for any agent i, there is some zero-measure set $\tilde{V}_i$ such that for any v_i in $[\underline{v}, \overline{v}]\backslash\tilde{V}_i$, following the equilibrium action $\sigma_i(v_i)$ yields a strictly higher payoff than following any other action.

In the next subsection, we propose a mechanism, called the *Equal Cost Sharing with Pivotal Reward* mechanism (or simply the *ECSPR mechanism*), which approximately attains

[10] Similarly, assuming that $\alpha \leq \frac{n-1}{n}$, another trivial equilibrium is that all agents vote for the mechanism (regardless of their valuations), and (under some suitably designed mechanism) the public good will always be provided.

first-best efficiency under a very natural strict Bayesian Nash equilibrium.

2.3. ECSPR mechanism

Let n^* be the smallest integer greater than or equal to αn. That is, if there are n^* or more than n^* yes votes, the mechanism will be approved. The ECSPR mechanism, denoted by $\{q^E, \{t_i^E\}_{i=1}^n\}$, is an indirect mechanism. It asks each agent to report either a high value, denoted by h, or a low value, denoted by l. That is, letting $M = \{h, l\}$, q^E is a mapping from M^n to $[0, 1]$, and t_i^E is a mapping from M^n to R. Since we require that the mechanism be anonymous, what matters for the value of q^E is the number of agents who report h and the number of agents who report l. In addition, what matters for the value of t_i^E is the number of agents who report h, the number of agents who report l, *and* whether agent i reports h or l. Letting x be the number of agents whose reported values are h (which implies that the number of agents whose reported values are l is $n - x$), Table 1 describes the value of q^E for every possible x, and the value of t_i^E for every possible x and every possible reported value of agent i (noting that the first column in Table 1 denotes the number of h reports, rather than the number of yes votes). The parameters $\eta_1 > 0$, $\eta_2 > 0$, and $p \in (0, 1)$ are given. We call η_1 the *pivotal reward*.

The ECSPR mechanism has two basic features. First, when the number of agents reporting high values is equal to or greater than $n^* + 1$, the public good is provided with probability one and the cost of the public good is shared equally by all agents. Second, when the number of agents who report high values is exactly n^*, the public good is provided with probability p. In addition, in this case, if the public good is provided, then any agent who reports h makes a payment less than the average cost, while any agent who reports l makes a payment more than the average cost.

Notice that for the ECSPR mechanism to work properly, we need to assume that $\alpha \leq (n-1)/n$. This is because if $\alpha > (n-1)/n$, then we have $n^* = n$, which implies that the budget cannot be balanced for the case in which there are n^* reported high values, because in

Table 1: Description of the ECSPR mechanism (conditional on approval of the mechanism).

x	t_i^E (if agent i reports h)	t_i^E (if agent i reports l)	q^E
0	N/A	0	0
. . .	0	0	0
n^*-1	$-\eta_2$	$\frac{n^*-1}{n-n^*+1}\eta_2$	0
n^*	$p(\frac{C}{n}-\eta_1)$	$p(\frac{C}{n}+\frac{n^*}{n-n^*}\eta_1)$	p
n^*+1	$\frac{C}{n}$	$\frac{C}{n}$	1
. . .	$\frac{C}{n}$	$\frac{C}{n}$	1
n	$\frac{C}{n}$	N/A	1

this case, all n agents will receive pivotal rewards, while no agent contributes to the rewards.

2.4. Equilibrium analysis

For any given $\eta_1 > 0$ and $p \leq \frac{n-n^*}{n}$, define $z(\eta_1, p)$ as the unique $z \in [0,1]$ that satisfies $z = 1 - F(\frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z}{z}+1-p})$.[11] Let $t(\eta_1,p) = \frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z(\eta_1,p)}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z(\eta_1,p)}{z(\eta_1,p)}+1-p}$ (so, $1 - F(t(\eta_1,p)) = z(\eta_1,p)$). If we set $t(\eta_1,p) = \underline{v}$, then $\eta_1 = (\frac{C}{n} - \underline{v})\frac{1-p}{p}\frac{n-n^*}{n^*}$ (using the fact that $z(\eta_1,p) = 1 - F(t(\eta_1,p)) = 1 - F(\underline{v}) = 1$). We let $\eta_1(p) = (\frac{C}{n} - \underline{v})\frac{1-p}{p}\frac{n-n^*}{n^*}$.

Lemma 1. *Assume that* $\alpha \leq \frac{n-1}{n}$, $p \leq \frac{n-n^*}{n}$ *and* $0 < \eta_1 < \eta_1(p)$. *In the voting game*

[11] The uniqueness of the solution z in the equation follows from the following facts: (i) the left-hand side of the equation is strictly increasing in z; (ii) the right-hand side of the equation is decreasing in z because $1 - F(\frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z}{z}+1-p}) = 1 - F(\frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1}{z}-p\frac{n^*}{n-n^*}+1-p}) = 1 - F(\frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1}{z}+1-p\frac{n}{n-n^*}}) = 1 - F(\frac{C}{n} - \frac{\frac{n^*}{n-n^*}p\eta_1}{p\frac{n^*}{n-n^*}+(1-p\frac{n}{n-n^*})z})$ and $1 - p\frac{n}{n-n^*} \geq 0$ (which is due to the fact that $p \leq \frac{n-n^*}{n}$); (iii) when z approaches zero, the right-hand side $1 - F(\frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z}{z}+1-p}) = 1 - F(\frac{C}{n} - \frac{\frac{n^*}{n-n^*}p\eta_1}{p\frac{n^*}{n-n^*}+(1-p\frac{n}{n-n^*})z})$ approaches $1 - F(\frac{C}{n} - \eta_1)$, which is greater than the left-hand side (i.e., 0); and (iv) when z approaches 1, the right-hand side approaches $1 - F(\frac{C}{n} - \frac{n^*}{n-n^*}\frac{p\eta_1}{1-p})$, which is less than the left-hand side (i.e., 1).

in which the principal proposes the ECSPR mechanism, there is a strict Bayesian Nash equilibrium in which an agent votes for the mechanism and reports h if the agent's valuation is in $[t(\eta_1, p), \overline{v}]$, and the agent votes against the mechanism and reports l if the agent's valuation is in $[\underline{v}, t(\eta_1, p))$.

Proof of Lemma 1:

We use "yes" to denote an agent's action of voting for the ECSPR mechanism and "no" to denote an agent's action of voting against the ECSPR mechanism. Since agents choose their voting decisions and report their values simultaneously, an agent has the following four actions to choose from: (yes, h), (yes, l), (no, h), (no, l).

Consider agent i, whose valuation is v_i. *Suppose that all agents other than i follow the $t(\eta_1, p)$-threshold strategy.* We will show that it is optimal for agent i to follow such a strategy.

Given that all agents other than i follow the $t(\eta_1, p)$-threshold strategy, there are four possibilities to consider:

(a) $n^* - 2$ or fewer others vote yes and report h. In this case, there is nothing that i can do to get the mechanism approved, and thus agent i is indifferent between the four actions.

(b) Exactly $n^* - 1$ others vote yes and report h. In this case, (no, h) and (no, l) both give a payoff of zero, while (yes, h) and (yes, l) give payoffs of $p(v_i - (\frac{C}{n} - \eta_1))$ and $-\frac{n^*-1}{n-n^*+1}\eta_2$, respectively.

(c) Exactly n^* others vote yes and report h. Then (no, h) and (yes, h) give payoff $v_i - \frac{C}{n}$, while (no, l) and (yes, l) give payoff $p(v_i - (\frac{C}{n} + \frac{n^*}{n-n^*}\eta_1))$.

(d) If $n^* + 1$ or more vote yes and report h, the announcement is irrelevant.

It is obvious that (yes, l) is strictly worse than (no, l) in expectation since the two actions give the same payoff except when $n^* - 1$ others vote yes and report h (in which case (yes, l) is strictly worse than (no, l) as $-\frac{n^*-1}{n-n^*+1}\eta_2 < 0$). So, (yes, l) cannot be a best reply for any $v_i \in [\underline{v}, \overline{v}]$.

We next show that (no, h) cannot be a best reply for almost any $v_i \in [\underline{v}, \overline{v}]$. In particular, (yes, h) and (no, h) only changes the outcome when $n^* - 1$ others vote yes and report h, so (yes, h) is weakly worse than (no, h) if

$$(1) \qquad p(v_i - (\frac{C}{n} - \eta_1)) \leq 0 \Leftrightarrow v_i - \frac{C}{n} + \eta_1 \leq 0.$$

Also, (no, l) is weakly worse than (no, h) if

$$(2) \qquad p(v_i - (\frac{C}{n} + \frac{n^*}{n - n^*}\eta_1)) \leq v_i - \frac{C}{n}.$$

Hence, if (no, h) is a best reply, it follows from (1) that $v_i \leq \frac{C}{n} - \eta_1$. But then, if $p < \frac{n-n^*}{n}$, since

$$v_i - (\frac{C}{n} + \frac{n^*}{n - n^*}\eta_1) < v_i - \frac{C}{n} < 0$$

(where the last inequality follows from the fact that $v_i \leq \frac{C}{n} - \eta_1 < \frac{C}{n}$), it follows that

$$(3) \qquad \begin{aligned} p(v_i - (\frac{C}{n} + \frac{n^*}{n - n^*}\eta_1)) >& \frac{n - n^*}{n}(v_i - (\frac{C}{n} + \frac{n^*}{n - n^*}\eta_1)) \\ =& \frac{n - n^*}{n}(v_i - \frac{C}{n}) - \frac{n^*}{n}\eta_1 \\ =& (v_i - \frac{C}{n}) - \frac{n^*}{n}(v_i - \frac{C}{n} + \eta_1) \\ \geq& v_i - \frac{C}{n} \end{aligned}$$

so (2) cannot hold. We thus have shown that (no, h) cannot be a best reply for any $v_i \in [\underline{v}, \overline{v}]$ if $p < \frac{n-n^*}{n}$.

If $p = \frac{n-n^*}{n}$, it can be verified that (i) if $v_i < \frac{C}{n} - \eta_1$, then the last inequality in (3) will be strict (although the first inequality in (3) will become an equality) and thus (2) cannot hold (which implies that (no, l) is strictly better than (no, h)); (ii) if $v_i > \frac{C}{n} - \eta_1$, then (1) cannot hold (which implies that (yes, h) is strictly better than (no, h)); and (iii) if $v_i = \frac{C}{n} - \eta_1$, then (1) and (2) will be equalities (which implies that i is indifferent between (no, h), (yes, h) and (no, l), all of which are actually optimal actions for i). So (no, h) cannot be a best reply for *almost* any type of agent i (in particular, for any $v_i \neq \frac{C}{n} - \eta_1$).

Hence, (yes, h) and (no, l) are the only possibilities when all others follow the $t(\eta_1, p)$-threshold strategy (except when $p = \frac{n-n^*}{n}$ *and* $v_i = \frac{C}{n} - \eta_1$, in which case i is indifferent between (no, h), (yes, h) and (no, l)). Let $z(\eta_1, p) = Prob(v_j \geq t(\eta_1, p))$ be the probability that agent $j \neq i$ chooses (yes, h) and

$$\rho(k) = \frac{(n-1)!}{k!(n-1-k)!} z^k (1-z)^{n-1-k} \tag{4}$$

be the probability that k others select (yes, h). Now, (yes, h) and (no, l) give different payoffs in two cases:

(a) When $n^* - 1$ others select (yes, h), in which case (yes, h) gives payoff $p(v_i - (\frac{C}{n} - \eta_1))$ and (no, l) gives 0.

(b) When n^* others select (yes, h), in which case (yes, h) gives payoff $v_i - \frac{C}{n}$ and (no, l) gives $p(v_i - (\frac{C}{n} + \frac{n^*}{n-n^*}\eta_1))$. Hence the difference is

$$\Delta = \rho(n^* - 1) p(v_i - \frac{C}{n} + \eta_1) + \rho(n^*)[v_i - \frac{C}{n} - p(v_i - (\frac{C}{n} + \frac{n^*}{n-n^*}\eta_1))].$$

Since

$$\begin{aligned}
\rho(n^* - 1) =& \frac{(n-1)!}{(n^*-1)!(n-n^*)!} z(\eta_1, p)^{n^*-1}(1 - z(\eta_1, p))^{n-n^*} \\
=& \frac{n^*(n-1)!}{n^*(n^*-1)!(n-n^*)!} z(\eta_1, p)^{n^*-1}(1 - z(\eta_1, p))^{n-n^*} \\
=& \frac{n^*}{n-n^*} \frac{1 - z(\eta_1, p)}{z(\eta_1, p)} \frac{(n-1)!}{n^*!(n-1-n^*)!} z(\eta_1, p)^{n^*}(1 - z(\eta_1, p))^{n-1-n^*} \\
=& \frac{n^*}{n-n^*} \frac{1 - z(\eta_1, p)}{z(\eta_1, p)} \rho(n^*)
\end{aligned}$$

it follows that $\Delta >$ (respectively $<$) 0 if and only if

$$\frac{n^*}{n-n^*} \frac{1 - z(\eta_1, p)}{z(\eta_1, p)} p(v_i - \frac{C}{n} + \eta_1) + v_i - \frac{C}{n} - p(v_i - (\frac{C}{n} + \frac{n^*}{n-n^*}\eta_1)) > \text{(respectively <) } 0,$$ [12]

[12] The "if and only if" argument holds if $\rho(n^*) > 0$, which is true because it can be shown that $0 <$

i.e.,

$$(v_i - \frac{C}{n})(1 + p\frac{n^*}{n-n^*}\frac{1-z(\eta_1,p)}{z(\eta_1,p)} - p) > \text{(respectively} <) \ -\frac{n^*}{n-n^*}\frac{p}{z(\eta_1,p)}\eta_1,$$

i.e.,

$$v_i > \text{(respectively} <) \ \frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z(\eta_1,p)}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z(\eta_1,p)}{z(\eta_1,p)} + 1 - p}.$$

We have thus shown that it is optimal for agent i to adopt the $t(\eta_1, p)$-threshold strategy, given that all other agents use the $t(\eta_1, p)$-threshold strategy. Thus, all agents using the $t(\eta_1, p)$-threshold strategy is a Bayesian Nash equilibrium. Furthermore, from the analysis above, it is also a strict Bayesian Nash equilibrium. □

Notice that η_2 does not appear in the threshold $t(\eta_1, p)$. This is because η_2 is used by the principal as a punishment for inconsistent strategies (in particular, it can prevent an agent from deviating to (yes, l) in the $t(\eta_1, p)$-threshold strategy equilibrium).[13] As long as η_2 is positive, this punishment will be effective, and the exact size of η_2 will not affect the agents' equilibrium behavior.

The $t(\eta_1, p)$-threshold strategy equilibrium may not be the unique strict Bayesian Nash equilibrium of the voting game (we will discuss other strict BNE in Section 2.7). However, if we restrict the equilibria of the game to the set of *symmetric consistent-strategy* equilibria, then the $t(\eta_1, p)$-threshold strategy is indeed unique. An equilibrium is *symmetric* if all agents use the same strategy. An agent's strategy is *consistent* if the agent votes for the mechanism if and only if the agent reports h (or equivalently, the agent votes against the mechanism if and only if the agent reports l). A consistent-strategy equilibrium does not restrict an agent's strategy space to the set of consistent strategies; it instead requires that when all other agents use consistent strategies, an agent's best response is a consistent strategy.

$z(\eta_1, p) < 1$ (under the assumption that $0 < \eta_1 < \eta_1(p)$). In particular, if $\eta_1 = 0$, then $t(\eta_1, p) = \frac{C}{n}$, and if $\eta_1 = \eta_1(p)$, then $t(\eta_1, p) = \underline{v}$. Using the fact that $t(\eta_1, p)$ is strictly decreasing in η_1 (see Lemma 3), we have $t(\eta_1, p) \in (\underline{v}, \frac{C}{n})$ when $0 < \eta_1 < \eta_1(p)$, which implies that $z(\eta_1, p) \in (0, 1)$.

[13] On the other hand, the feature that the public good is provided with a probability less than one in the case where the number of h reports is n^* can prevent an agent from deviating to (no, h) in the $t(\eta_1, p)$-threshold strategy equilibrium.

Lemma 2. *Assume that* $\alpha \le \frac{n-1}{n}$, $p \le \frac{n-n^*}{n}$ *and* $0 < \eta_1 < \eta_1(p)$. *In the voting game in which the principal proposes the ECSPR mechanism, the* $t(\eta_1, p)$*-threshold strategy equilibrium is the unique symmetric consistent-strategy strict Bayesian Nash equilibrium.*

In the remainder of the paper (except Section 2.7), we will focus on the $t(\eta_1, p)$-threshold strategy equilibrium. In such a consistent-strategy equilibrium, there is a clear link between agents' voting behavior and their reported values. This makes the equilibrium easy to analyze.

We next analyze the relationship between $t(\eta_1, p)$ and η_1 and p. The fact that the pivotal reward $\eta_1 > 0$ does not automatically mean that in equilibrium, all agents will vote yes for the mechanism. This is because when an agent's vote is pivotal, voting yes (and reporting h) will cause the public good to be provided with some probability, while voting no will result in no provision of the public good. So, if an agent has a very low valuation, then he may choose to vote no. However, we can show that as η_1 increases, the fraction of agents who choose to vote no will decrease. This result is illustrated in the next lemma. In addition, the lemma shows that as p increases, the threshold $t(\eta_1, p)$ decreases.

Lemma 3. *Assume that* $p \le \frac{n-n^*}{n}$. *Then* $t(\eta_1, p)$ *is strictly decreasing in* η_1 *and* p.

Recall that if we set $t(\eta_1, p)$ to be $\underline{v}$, then we have $\underline{v} = \frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{1-F(\underline{v})}\eta_1}{p\frac{n^*}{n-n^*}\frac{F(\underline{v})}{1-F(\underline{v})}+1-p} = \frac{C}{n} - \frac{\frac{n^*}{n-n^*}p\eta_1}{1-p}$, i.e., $\eta_1 = (\frac{C}{n} - \underline{v})\frac{1-p}{p}\frac{n-n^*}{n^*} = \eta_1(p)$. This implies that for any given $p \in (0, \frac{n-n^*}{n}]$, we can set η_1 to be sufficiently close to $\eta_1(p)$ so that $t(\eta_1, p)$ is sufficiently close to $\underline{v}$ and thus $z(\eta_1, p)$ is sufficiently close to one. Therefore, the probability that an agent will vote for the mechanism and report high value is sufficiently close to one, and the probability that the public good will be provided is sufficiently close to one.[14] We thus have the following result.

Proposition 1. *Assume that* $\alpha \le \frac{n-1}{n}$ *and* $p \le \frac{n-n^*}{n}$. *Suppose that the principal proposes the ECSPR mechanism and that agents use the* $t(\eta_1, p)$*-threshold strategy in the voting game.*

[14]Notice that we cannot let $\eta_1 = \eta_1(p)$ (or $\eta_1 > \eta_1(p)$), because if this is the case, then the $t(\eta_1, p)$-threshold strategy equilibrium is such that all agents vote for the mechanism. But such an equilibrium is not a strict Bayesian Nash equilibrium, because any agent is indifferent between voting for the mechanism and voting against it (as long as $\alpha \le (n-1)/n$).

Then for any $\zeta > 0$, there exists a pivotal reward $\eta_1 > 0$ such that the probability of provision of the public good in equilibrium is greater than $1 - \zeta$.

According to Lemma 3, $t(\eta_1, p)$ is strictly decreasing in p. So, if we set p to its maximum possible value, $\frac{n-n^*}{n}$, then the ECSPR mechanism will be supported by the maximum fraction of agents. In addition, if we set $p = \frac{n-n^*}{n}$, then $t(\eta_1, p) = \frac{C}{n} - \eta_1$. In the remainder of the paper, we will set $p = \frac{n-n^*}{n} := p^*$. With some slight abuse of notation, we will simply write $t(\eta_1, p^*)$ as $t(\eta_1)$ and $z(\eta_1, p^*)$ as $z(\eta_1)$.

2.5. Large economy

This subsection considers a large economy. For simplicity, throughout this subsection, we assume that $\frac{C}{n} = c$ for some constant c for any n.

To get the ECSPR mechanism approved, we do not need to have almost all agents vote for the mechanism; it is sufficient that more than α fraction of agents choose to vote for the mechanism. To accomplish this in a large economy, the pivotal reward η_1 should be strictly greater than η_1^m, where η_1^m is such that $1 - F(t(\eta_1^m)) = \alpha$. That is, η_1 should be strictly greater than $c - F^{-1}(1-\alpha)$ (because $t(\eta_1) = \frac{C}{n} - \eta_1 = c - \eta_1$). We call the term $c - F^{-1}(1-\alpha)$ the *lower bound of pivotal reward*. On the other hand, the term $\frac{C}{n} - \underline{v} = c - \underline{v}$ measures the amount of the reward the principal needs to give a pivotal agent to induce the lowest valuation agent to vote for the mechanism. We call $c - \underline{v}$ the *upper bound of pivotal reward*.

If the pivotal reward is strictly between the lower bound and the upper bound of pivotal reward, then there is a positive probability that the ECSPR mechanism will be rejected by agents when the economy is small. However, as the economy becomes large, the probability that the mechanism will be approved goes to one. Intuitively, this is due to the law of large numbers, which ensures that the approval rate of the mechanism converges to $z(\eta_1)$, a value that is greater than α. We thus obtain the following result.

Theorem 1. *Assume that $\alpha \leq \frac{n-1}{n}$ and $\frac{C}{n} = c$ for any n. Suppose that the mechanism proposed by the principal is the ECSPR mechanism with $p = p^*$, and that agents use the*

$t(\eta_1)$-threshold strategy in the voting game. If $\eta_1 \in (c - F^{-1}(1-\alpha), c - \underline{v})$, then the probability of provision of the public good in equilibrium goes to one as n goes to infinity.

We call any pivotal reward that is strictly between the lower bound and the upper bound of pivotal reward the *effective pivotal reward*. A careful analysis of the effective pivotal reward will lead to several interesting observations. First, the range of the effective pivotal reward is independent of n, because the lower bound and the upper bound of pivotal reward are independent of n. This also implies that the punishment for an agent who votes no in the event that the mechanism is just barely approved is also independent of n (more precisely, the range of the punishment per agent, $\frac{n^*}{n-n^*}\eta_1$, is $(\frac{\alpha}{1-\alpha}(c - F^{-1}(1-\alpha)), \frac{\alpha}{1-\alpha}(c - \underline{v})))$. This property is nice, because it means that the punishment is bounded even in a large economy, and this makes our mechanism a practical one. Second, as α changes from $1 - F(c)$ to 1,[15] the range of the effective pivotal reward becomes smaller and smaller. This reflects the fact that as α increases, it becomes harder to design a mechanism to achieve first-best efficiency.

We now compare our result with that of Mailath and Postlewaite (1990). Mailath and Postlewaite (1990) show that if the individual rationality constraint is required for all agents, then as the economy becomes large, the probability of provision of the public good goes to zero for *any* sequence of mechanisms. This implies that in our voting game, if the voting rule is a unanimity rule, then the probability of provision of the public good goes to zero as the economy becomes large (see the proof of Corollary 1 (i) for a detailed analysis). On the other hand, our result (Theorem 1) shows that as long as the voting rule is *not* the unanimity rule (noticing that $(n-1)/n$ approaches 1 as n becomes large), then first-best efficiency can be achieved approximately. We use $p(\mathcal{M})$ to denote the maximum probability (under all BNEs) that the public good will be provided in the voting game when the mechanism proposed by the principal is $\mathcal{M}$. We thus have:

Corollary 1.

[15] If $\alpha < 1 - F(c)$, then the principal can simply propose the equal cost sharing mechanism, which will be approved by agents under the α-majority rule in a large economy.

(i) If $\alpha = 1$, then for any sequence of mechanisms $\{\mathcal{M}^n\}_{n=1}^{\infty}$ where $\mathcal{M}^n$ denotes a mechanism for the n-agent economy, we have that $p(\mathcal{M}^n) \to 0$ as $n \to \infty$.

(ii) If $\alpha < 1$, then there exists a sequence of mechanisms $\{\mathcal{M}^n\}_{n=1}^{\infty}$ such that $p(\mathcal{M}^n) \to 1$ as $n \to \infty$.

It is worth emphasizing the difference and the link between setting $\alpha = 1$ and the requirement of (IR) (as in Mailath and Postlewaite 1990). In particular, setting $\alpha = 1$ requires that in order for a mechanism to be approved, any agent must obtain a non-negative payoff by voting for the mechanism, *conditional on the event that his vote is pivotal* (i.e., *conditional on the event that all other agents vote for the mechanism*), while (IR) requires that any agent obtain a non-negative payoff from implementation of the mechanism. So, although both requirements require an agent's payoff to be non-negative, the payoff in the former case is a conditional expected payoff, while the payoff in the latter case is an unconditional expected payoff. Despite this difference, there is a very close link between setting $\alpha = 1$ and (IR). In particular, in the proof of Corollary 1, we show that for any budget-balanced mechanism $\mathcal{M}_1$ proposed by the principal and any BNE of our voting game (with $\alpha = 1$), we can always find a slightly different mechanism $\mathcal{M}_2$ such that $\mathcal{M}_2$ satisfies (IC), (BB), and (IR).[16] In addition, $\mathcal{M}_2$ and $\mathcal{M}_1$ are equivalent in the sense that they yield the same probability of provision in equilibrium. So, Corollary 1 (i) immediately follows from the impossibility result of Mailath and Postlewaite (1990).

2.6. An example

Example 1. Assume that the agents' valuations are i.i.d. with a uniform distribution on $[0, 1]$. Assume that $\frac{C}{n} = c = 0.4$, and $\alpha = \frac{2}{3}$. Then the *lower bound of pivotal reward* per agent is $c - F^{-1}(1 - \alpha) = 0.067$, the *upper bound of pivotal reward* per agent is $c - \underline{v} = 0.4$,

[16]In particular, $\mathcal{M}_2$ is constructed as follows. If the valuation profile $v = (v_1, v_2, ..., v_n)$ is such that some agent chooses to vote against the mechanism $\mathcal{M}_1$ in equilibrium, then $\mathcal{M}_2$ is such that the public good is not provided and no payment is collected from any agent. Otherwise (i.e., if v is such that all agents choose to vote for $\mathcal{M}_1$), then $\mathcal{M}_2$ yields the same probability of provision and payment scheme as $\mathcal{M}_1$.

and the threshold valuation $t(\eta_1) = 0.4 - \eta_1$.

Figure 1 depicts the probability of provision in the voting game as a function of the pivotal reward for $n = 100, 500, 1000, 10000$. The figure makes it clear that as the economy becomes large, the probability of provision goes to one when the pivotal reward is strictly between the lower bound and upper bound of pivotal reward. In addition, for any given size of the economy, the figure shows that as the pivotal reward increases, the probability of provision increases. So, there is a monotonic relationship between the pivotal reward and the probability of provision of the public good.

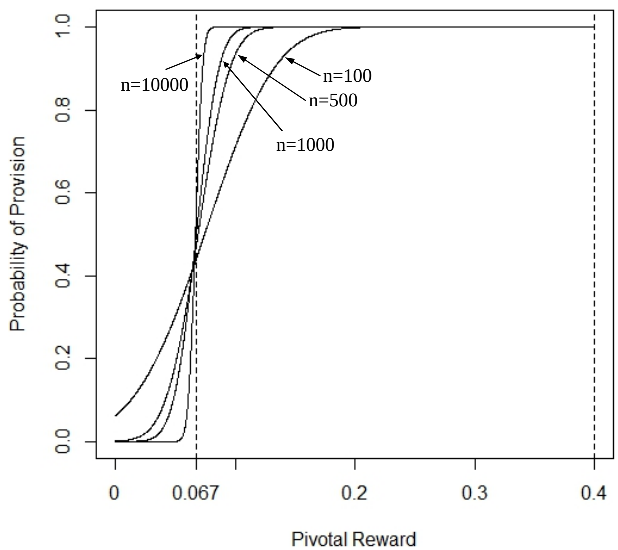

Figure 1: Probability of provision vs pivotal reward in Example 1.

Finally, we want to emphasize that the fact that a mechanism is supported by an agent does not necessarily mean that the agent obtains a payoff greater than zero from implementation of the mechanism. In Example 1, the average cost of the public good is 0.4, and according to our mechanism, the cost is equally distributed among agents when the public good is provided (except for the case where there are some pivotal agents, which occurs with probability zero in a large economy). This implies that around 40% of the agents are hurt if the mechanism is implemented. In other words, the individual rationality constraints of around 40% of the agents are violated under the mechanism, although more than 2/3 of agents will choose to vote for the mechanism (provided that the pivotal reward is in between the lower bound and the upper bound of pivotal reward).

2.7. Other equilibria and equilibrium selection

The previous sections focus on the $t(\eta_1)$-threshold strategy BNE. In this subsection, we will discuss the properties of other strict BNE and explain why the $t(\eta_1)$-threshold strategy BNE is a natural choice of equilibrium. Since $t(\eta_1)$-threshold strategy BNE is the unique symmetric consistent-strategy strict BNE (Lemma 2), any symmetric strict BNE that is different from the $t(\eta_1)$-threshold strategy BNE must involve the use of inconsistent strategies. We say that a strict BNE is an *inconsistent-strategy strict BNE* if it is not a consistent-strategy strict BNE. We have the following result.

Lemma 4. *Assume that $\alpha \leq \frac{n-1}{n}$ and $p = p^*$. In any symmetric inconsistent-strategy strict BNE of the voting game in which the principal proposes the ECSPR mechanism, we must have:*

(i) agent i votes against the mechanism and reports l if $v_i \leq \frac{C}{n} - \eta_1$;

(ii) agent i votes for the mechanism and reports h if $v_i \geq \frac{C}{n}$;

(iii) there exists a set $\hat{V} \subset (\frac{C}{n} - \eta_1, \frac{C}{n})$ (where $\hat{V}$ has a non-zero measure) such that if $v_i \in \hat{V}$, then agent i votes against the mechanism and reports h, and if $v_i \in (\frac{C}{n} - \eta_1, \frac{C}{n})\backslash\hat{V}$, then agent i votes for the mechanism and reports h.

Note that Lemma 4 shows that in any symmetric inconsistent-strategy strict BNE, an agent will never use the inconsistent action (yes, l). The other inconsistent action (no, h) may be used only when $\frac{C}{n} - \eta_1 < v_i < \frac{C}{n}$. An implication of Lemma 4 is that when agents use a symmetric inconsistent-strategy strict BNE, the ECSPR mechanism will be approved with a lower probability and thus the public good will be provided with a lower probability than the case in which agents use the $t(\eta_1)$-threshold strategy BNE (noting that $t(\eta_1) = \frac{C}{n} - \eta_1$). Another implication of Lemma 4 is that in any symmetric inconsistent-strategy strict BNE, the strategy used by an agent must have at least two thresholds, because as v_i varies from $\underline{v}$

to $\overline{v}$, three equilibrium actions (i.e., (no, l), (no, h), (yes, h)) occur. Thus, the $t(\eta_1)$-threshold strategy BNE—which involves only one threshold in its equilibrium strategy—is the most simple/salient BNE among all symmetric strict BNE.

Finally, note that the $t(\eta_1)$-threshold strategy BNE achieves efficiency in a large economy, which implies that the total expected surplus of all agents reaches its maximum. This in turn implies that the $t(\eta_1)$-threshold strategy BNE (weakly) Pareto dominates all other symmetric inconsistent-strategy strict BNE in a large economy in the ex ante sense, i.e., an agent's ex ante payoff under the $t(\eta_1)$-threshold strategy BNE is (weakly) higher than the agent's ex ante payoff under any symmetric inconsistent-strategy strict BNE (noting that comparing agents' ex ante payoffs across equilibira is relevant if agents decide which strategies to use *before* they know their (private) valuations).

3. Extension: Common Value Component

3.1. Setup

In the previous section, we assume that the principal knows the distribution of agents' valuations. This implies that in a large economy, even without knowing agents' private valuations, the principal knows whether the public good should be provided or not. This makes the voting stage of our model less interesting, since the voting stage only serves as a device for getting the public good provided (if the public good should be provided), rather than a device that the principal uses to collect useful valuation information from agents to help the principal to make the decision about provision of the public good.

In this section, we will relax the assumption that the principal knows the exact distribution of agents' valuations. Following the common value component model of Ledyard and Palfrey (2002), we assume that the agents' valuations have a *common value component*. More specifically, an agent's (adjusted) valuation is $r_i = v_i + m$, where v_i is agent i's private valuation and m is the common value component. We assume that $v_1, ..., v_n$ are i.i.d. with a continuous distribution function F, and m takes a value in the range of $[\underline{m}, \overline{m}]$. All agents know the exact value of m, but the principal only knows the distribution of m. This implies that an agent knows the exact distribution of any other agent's valuation, but the principal does not know the exact distribution of the valuation of any agent. We assume throughout this section that the total cost of the public good is $C(n) = nc$ for any n, where c is a constant.

3.2. Analysis

Define $m^* = c - v^e$, where v^e is the expected value of v_i. Obviously, if $m > m^*$, then $v^e + m > c$ and thus the public good *should* be provided in a large economy. If $m < m^*$, then $v^e + m < c$ and thus the public good *should not* be provided in a large economy. We assume that $\underline{m} < m^* < \overline{m}$, which implies that both provision and non-provision of the public good may be optimal, depending on the realized value of m.

Suppose that the principle proposes the ECSPR mechanism with $p = p^* = \frac{n-n^*}{n}$. Note that from the point of view of the agents playing the game induced by the mechanism, it is the same as with independent private values (except that there is a common "shift" of the support of agents' valuations). Let $t(\eta_1|m)$ be the threshold $t(\eta_1)$ that we define in Section 2.4 when the realization of the common value component is m, and $z(\eta_1|m)$ be the probability that an agent's valuation is greater than $t(\eta_1|m)$. Let η_1^* be such that $z(\eta_1^*|m^*) = \alpha$. That is, η_1^* is the pivotal reward that induces just α fraction of agents to vote for the mechanism in a large economy in the $t(\eta_1|m)$-threshold strategy equilibrium when $m = m^*$. We have the following result regarding the relationship between $z(\eta_1|m)$ and m.

Lemma 5. *$z(\eta_1|m)$ is increasing in m. In addition, $z(\eta_1|m)$ is strictly increasing in m at any m where $z(\eta_1|m) \in (0, 1)$.*

Note that $z(\eta_1^*|m^*) = \alpha \in (0, 1)$. According to Lemma 5, if $m > m^*$, then $z(\eta_1^*|m) > \alpha$, and if $m < m^*$, then $z(\eta_1^*|m) < \alpha$. So, if the principal proposes the ECSPR mechanism with

$\eta_1 = \eta_1^*$ and agents use the $t(\eta_1^*|m)$-threshold strategy equilibrium, then (i) if $m > m^*$, the mechanism will be approved with probability one (and the public good will be provided with probability one) in a large economy; and (ii) if $m < m^*$, the mechanism will not be approved (and the public good will not be provided) in a large economy. That is, the first-best efficient level of provision of the public good can always be achieved in a large economy. This result is summarized in the following theorem.

Theorem 2. *Assume that $\alpha \leq \frac{n-1}{n}$. Suppose that there is a common value component m in an agent's valuation. Suppose that the mechanism proposed by the principal is the ECSPR mechanism with $p = p^*$ and $\eta_1 = \eta_1^*$, and that agents use the $t(\eta_1^*|m)$-threshold strategy in the voting game. We have:*

(i) if $m > m^*$, then the probability of provision of the public good in equilibrium goes to one as n goes to infinity;

(ii) if $m < m^*$, then the probability of provision of the public good in equilibrium goes to zero as n goes to infinity.

4. Choice of Voting Rule

An implication of the results in the previous sections is that the choice of voting rule is *irrelevant* with respect to the efficiency of provision of the public good, as long as the voting rule is not the unanimity rule. This is because as long as the proposed mechanism is the ECSPR mechanism (with a properly set η_1), then it can "rectify" any predetermined (but usually too stringent or too weak) voting rule, so that first-best efficiency of provision of the public good is always achieved. In this section, we consider the situations or constraints that may prevent the proper functioning of the ECSPR mechanism. In such situations, the initial choice of the voting rule becomes important. In this section, the voting rule is no longer exogenously given; instead, there is a constitutional design stage prior to the voting game. Our question is: Which voting rule should the principal choose at the constitutional design stage? In particular, when will the simple-majority rule be optimal, and when will a super-majority rule be optimal?

The analysis in this section is based on the model developed in Section 3. In addition, we assume that $\frac{1}{2} \leq \alpha \leq \frac{n-1}{n}$ (we require $\alpha \geq \frac{1}{2}$ because we seldom observe a voting rule with $\alpha < \frac{1}{2}$ in reality, and we require $\alpha \leq \frac{n-1}{n}$ because the efficiency result will never hold in the case of $\alpha = 1$).

4.1. No discriminatory mechanism

We first consider the situation in which the principal cannot use any discriminatory mechanism, i.e., the principal cannot charge different agents with different payments. This situation is likely to arise if the principal has fairness concerns, or it is required by law that the principal cannot use any discriminatory mechanism, or it is simply too costly to implement a discriminatory mechanism. In such situations, the only mechanism the principal can use is the *Equal Cost Sharing* mechanism (or simply the ECS mechanism). The ECS mechanism, $\{q^{ECS}, \{t_i^{ECS}\}_{i=1}^n\}$, is such that $q^{ECS}(v) = 1$ and $t_i^{ECS}(v) = \frac{C(n)}{n} = c$ for *any* $v \in [\underline{v}, \overline{v}]^n$. If the principal proposes the ECS mechanism, then an agent will vote for the mechanism if and only if the agent's valuation is above the average cost.[17] This is because (i) the ECS mechanism, once approved, will provide the public good with probability one and require each agent to pay the average cost of the public good, regardless of agents' reported valuations, and (ii) for any agent, voting for the ECS mechanism will induce it to be approved with a higher probability (and thus the public good will be provided with a higher probability) than voting against it.

Notice that as the common value component m increases, the approval rate under the ECS mechanism also increases, because the approval rate under the ECS mechanism $P(r_i \geq$

[17] Note that letting agents vote on the ECS mechanism is equivalent to the following two-step procedure: First, let the agents vote on whether the public good should be provided, and second, if the voting result is such that the public good is provided (i.e., there are enough yes votes in the first step), then the cost is equally distributed among all agents. This two-step procedure and its welfare property are also studied by Ledyard and Palfrey (1994) and Ledyard and Palfrey (2002), in which it is called the *referendum with equal cost shares.*

$c) = 1 - F_m(c) = 1 - F(c - m)$ is an increasing function of m (where F_m is the cdf of r_i and F is the cdf of v_i). If α is too large (so that $\alpha > 1 - F(c - m^*)$), then the public good will be under-provided (in particular, for m that is greater than but close to m^*, the public good should be provided—but it will not be provided, as the ECS mechanism cannot be approved). If α is too small (so that $\alpha < 1 - F(c - m^*)$), then the public good will be over-provided (in particular, for m that is less than but close to m^*, the public good should not be provided—but it will be provided, as the ECS mechanism will be approved). Only if $\alpha = 1 - F(c - m^*) := \alpha^*$, will the first-best provision level of the public good be achieved for *any* possible realization of m. So, the α^*-majority voting rule is the *optimal* voting rule. Notice that $\alpha^* = 1 - F(c - m^*) = 1 - F(v^e)$.

If the distribution F is such that its mean equals to its median, then $\alpha^* = 1 - F(v^e) = \frac{1}{2}$. If the distribution F is such that the mean is greater than the median, then $\alpha^* < \frac{1}{2}$. Since we require $\alpha \geq \frac{1}{2}$, the constrained-optimal voting rule will be $\alpha^* = \frac{1}{2}$. Notice that most widely used distributions are such that the mean is equal to the median or greater than the median. Examples of the former case include uniform distribution, normal distribution, t distribution (when the mean is well defined), etc. Examples of the latter case include log-normal distribution, Chi-squared distribution, exponential distribution, etc.[18] In summary, we have the following result, which provides a rationale for the popularity of the simple-majority rule in practice.

Proposition 2. *Suppose that the principal can only use the ECS mechanism. Suppose that there is a common value component in an agent's valuation, and that an agent's private valuation follows a distribution whose mean is equal to or greater than the median. Then the simple-majority rule is optimal (in the sense that the maximum efficiency of provision of the public good is achieved).*

[18]The distributions in the former case are usually symmetric, which implies that the fraction of agents who have high valuations and the fraction of agents who have low valuations are equal. The distributions in the latter case usually have a long right tail, which implies that most agents have relatively low valuations, but a few agents may have very high valuations.

If F is such that the mean of v_i is less than the median of v_i, then $\alpha^* = 1 - F(v^e) > 1/2$, i.e., a super-majority rule is optimal. Such a distribution usually has a long left tail, which implies that most agents have high valuations and very few agents have low valuations. In reality, such distributions are quite rare.

Proposition 3. *Suppose that the principal can only use the ECS mechanism. Suppose that there is a common value component in an agent's valuation, and that an agent's private valuation follows a distribution whose means is less than the median. Then the optimal voting rule will be a super-majority rule.*

4.2. Vote buying and the influence of interest groups

Voting is usually influenced by interest groups. Some interest groups, such as the developer of a public project, might want the public good to be provided, regardless of realization of the common value component. Some other interest groups, such as environmentalists, may hope that the public good is not provided, regardless of realization of the common value component. We call any interest group that always wants the public good to be provided a *type I interest group*, while any interest group that always wants the public good not to be provided a *type II interest group*. An interest group may influence the collective decision of agents by "buying" votes. That is, the interest group can bribe agents to make them to vote in favor of the interest group's preference. A natural question is, what kind of voting rule exhibits the greatest degree of immunity to the intervention of an interest group?

Suppose that m^* is such that $\underline{m} < m^* < \overline{m}$ (so that both provision and non-provision of the public good may be optimal). Suppose that the principal proposes the ECSPR mechanism with $p = p^* = \frac{n-n^*}{n}$ and $\eta_1 = \eta_1^*$, and that agents use the $t(\eta_1^*|m)$-threshold strategy equilibrium, in which an agent will choose either (yes, h) or (no, l) in equilibrium. A type I interest group will thus buy votes so that some agents who originally chose (no, l) will now choose (yes, h). A type II interest group will buy votes so that some agents who originally chose (yes, h) will now choose (no, l).

We first consider the intervention of a type I interest group. We will calculate the total payments that this type I interest group needs to give agents to induce α-fraction of agents to vote for the ECSPR mechanism. For simplicity, we assume that the interest group knows the realization of m. This implies that a type I interest group will intervene only when $m < m^* = c - v^e$, because otherwise (i.e., $m > m^*$), the public good will be provided anyway (in a large economy). However, we assume that the interest group does not know each agent's valuation, which implies that the agents who have sufficiently high valuations (and will vote yes even if there is no bribe) may mimic as low-valuation agents and take the bribes. So, the interest group has to bribe at least n^* agents to get the mechanism approved.[19] If we rank the agents' valuations from the highest to the lowest, then the bribe that the interest group makes to an agent should be sufficiently large so that the agent with the n^*-th highest valuation is just better off by voting yes than voting no. In a large economy, the n^*-th highest valuation must be approximately $m + F^{-1}(1-\alpha) := r_i^*$, where F is the distribution function of an agent's private valuation. For simplicity, we assume that $\frac{C}{n} = c$ for any n. From the proof of Lemma 1, it is clear that for an agent with valuation r_i^*, the expected payoff of choosing (no, l) is higher than the expected payoff of choosing (yes, h) by the amount $-p_1 \times p \times (r_i^* - c + \eta_1^*) - p_2 \times (r_i^* - c) + p_2 \times p \times (r_i^* - (c + \frac{n^*}{n-n^*}\eta_1^*)) := P$, where $p_1 = \rho(n^* - 1) = Prob(x_{-i} = n^* - 1)$ and $p_2 = \rho(n^*) = Prob(x_{-i} = n^*)$, where x_{-i} is the number of yes votes among agents other than i. Using the facts that $p = p^* = \frac{n-n^*}{n} \approx 1 - \alpha$ and $\frac{p_1}{p_2} = \frac{n^*}{n-n^*}\frac{1-z}{z} \approx 1$ (where the second equality follows from the fact that z, the probability that an agent chooses to vote yes, is equal to α if the interest group's bribes are such that α-fraction of agents vote yes) and the fact that $\eta_1^* = v^e - F^{-1}(1-\alpha)$,[20] we have $P \approx -p_1 \times (1-\alpha) \times (r_i^* - c + \eta_1^*) - p_2 \times (r_i^* - c) + p_2 \times (1-\alpha) \times (r_i^* - (c + \frac{\alpha}{1-\alpha}\eta_1^*)) = (p_1(1-\alpha) + p_2\alpha)(c - \eta_1^* - r_i^*) = p_1(c - \eta_1^* - r_i^*) = p_1(m^* - m)$.

[19] We assume that the interest group can monitor an agent' voting decision and reported value if the agent takes the bribe. This implies that an agent who originally votes no cannot take the bribe and still vote no (or vote yes but report l).

[20] More precisely, notice that η_1^* is such that $Prob(r_i \geq t(\eta_1^*|m)|m = m^*) = \alpha$. That is, $Prob(v_i + m^* \geq c - \eta_1^*) = Prob(v_i \geq v^e - \eta_1^*) = \alpha$, where the first equality follows from the fact that $m^* = c - v^e$. So, we have $\eta_1^* = v^e - F^{-1}(1-\alpha)$.

Using the analysis above, the total bribes that a type I interest group needs to give agents is thus $n^*P \approx \alpha nP \approx \alpha np_1(m^* - m)$. Notice that $p_1 = Prob(x_{-i} = n^* - 1) = \frac{(n-1)!}{(n^*-1)!(n-n^*)!}(\alpha)^{n^*-1}(1-\alpha)^{n-n^*}$ (where $n^* \approx n\alpha$) is the probability that an agent's vote is pivotal. It can be verified that as α increases, p_1 increases, and when $\alpha = 1$, p_1 reaches its maximum, which is equal to one.[21] This implies that as α increases, the interest group must pay more to any single agent because the pivotal probability of any agent will increase. We call this effect the *pivotal probability effect*. On the other hand, as α increases, the interest group must bribe more agents. We call this effect the *size effect*. These two effects respond to the change in α in the same direction, and thus mutually reinforce each other. This implies that any super-majority rule is better than the simple-majority rule if the principal's main goal is to prevent the intervention of a type I interest group, because a super-majority rule will incur a larger cost for the interest group to successfully intervene than in the case of the simple-majority rule. Actually, the optimal voting rule is such that $\alpha = \frac{n-1}{n}$.

We will now consider a type II interest group.[22] The type II interest group only needs to intervene when $m > m^* = c - v^e$. In this case, the interest group needs to bribe at least $1-\alpha$ fraction of agents to ensure that the mechanism will not be approved. Using an analysis similar to the case of a type I interest group, it can be verified that the total bribes that a type II interest group needs to give agents is $(n - n^*)p_1(m - m^*) \approx (1-\alpha)np_1(m - m^*)$. As α increases, p_1 increases, but $1-\alpha$ decreases. So, the pivotal probability effect and the size effect respond to the change in α in opposite directions. Figure 2 depicts $(1-\alpha)p_1$ as a function of α for the case in which $n = 1000$. It is obvious that as α_1 increases, $(1-\alpha)p_1$ decreases.[23] This implies that the size effect dominates the pivotal probability effect, and

[21] In particular, it can be verified that when n is large, p_1 is approximately proportional to $\frac{1}{\sqrt{\alpha(1-\alpha)}}$, which is increasing in α as long as $\alpha \geq \frac{1}{2}$ (see also footnote 23 for a more detailed analysis).

[22] Similar to the case of a type I interest group, we assume that the type II interest group can monitor both an agent' voting decision and his reported value if the agent takes the bribe (see also footnote 19).

[23] The declining relationship holds for any n when n is large. In particular, note that x_{-i}—the number of yes votes among agents other than i—follows a binomial distribution $B(n-1, \alpha)$, which can be approximated by a normal distribution with expected value $(n-1)\alpha$ and variance $(n-1)\alpha(1-\alpha)$. So, for any $0 < \alpha < 1$, we have $p_1 = Prob(x_{-i} = n^* - 1) = Prob(x_{-i} \leq n^* - 1) - Prob(x_{-i} \leq n^* - 2) = Prob(\frac{n^*-2-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}} < \frac{x_{-i}-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}} \leq \frac{n^*-1-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}}) \approx \Phi(\frac{n^*-1-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}}) -$

thus the simple-majority rule is better than any super-majority rule if the principal's main goal is to prevent the intervention of a type II interest group.

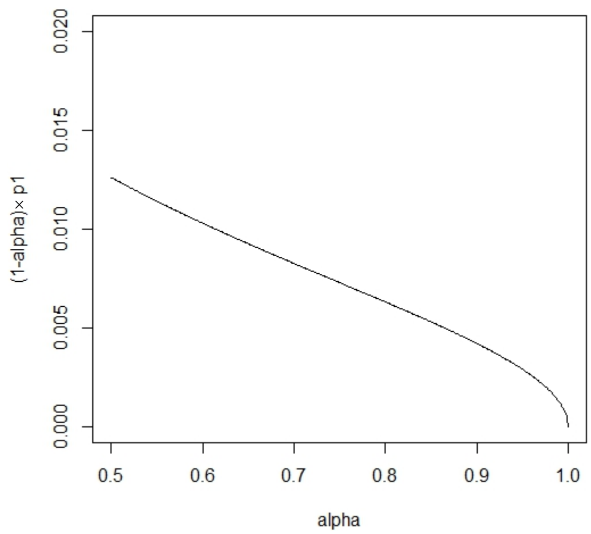

Figure 2: $(1-\alpha)p_1$ versus α for the case of $n = 1000$.

We thus have the following result.

Proposition 4. *If the principal wants to prevent the intervention of a type I interest group, then the principal should choose a super-majority voting rule. If the principal wants to prevent the intervention of a type II interest group, then the simple-majority rule is optimal.*

5. Conclusion

This paper considers the public good provision problem with a constitutional constraint, in which a public good provision mechanism can be used among a group of agents if and only if the mechanism can be approved by the agents under a prespecified α-majority voting rule. We find that if the principal proposes the so-called ECSPR mechanism, then as long

$\Phi(\frac{n^*-2-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}}) \approx \phi(\frac{n^*-1-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}}) \times (\frac{n^*-1-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}} - \frac{n^*-2-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}}) = \phi(\frac{n^*-1-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}})\frac{1}{\sqrt{(n-1)\alpha(1-\alpha)}} \approx \phi(\frac{n\alpha-1-(n-1)\alpha}{\sqrt{(n-1)\alpha(1-\alpha)}})\frac{1}{\sqrt{(n-1)\alpha(1-\alpha)}} = \phi(\frac{\alpha-1}{\sqrt{(n-1)\alpha(1-\alpha)}})\frac{1}{\sqrt{(n-1)\alpha(1-\alpha)}} \approx \phi(0)\frac{1}{\sqrt{(n-1)\alpha(1-\alpha)}}$, where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and pdf of a standard normal distribution, respectively. So, $p_1(1-\alpha) \approx \phi(0)\frac{1}{\sqrt{(n-1)\alpha(1-\alpha)}}(1-\alpha) = \phi(0)\frac{1}{\sqrt{(n-1)}}\sqrt{\frac{1-\alpha}{\alpha}}$ is decreasing in α for any given n (when n is large).

as $\alpha < 1$, first-best efficiency can always be achieved in a large economy. This is the case regardless of whether the agents' valuation are purely private valuations or have a common value component.

It should be noted that a more common mechanism in practice is probably the Equal Cost Sharing mechanism. Such a mechanism may not achieve full efficiency of provision of the public good, because it cannot adjust to the exogenously given constitutional constraint and thus leads to either over-provision or under-provision of the public good. Our paper thus demonstrates that if the principal can use discriminatory mechanisms (in which the probability of provision of the public good and the payment scheme depend on the agents' (reported) valuations)—such as the ECSPR mechanism—then the exogenously given constitutional constraint can be "corrected" and first-best efficiency can be achieved.[24]

It should also be noted that the efficiency result we obtain relies on the equilibrium we use—in particular, the $t(\eta_1)$-threshold strategy BNE. Although the $t(\eta_1)$-threshold strategy BNE is a natural equilibrium selection in terms of simplicity and efficiency (as we analyze in Section 2.7), it is not the unique strict BNE of the voting game. In this sense, what our result suggests is a *possibility* of achieving efficiency when $\alpha < 1$. This is in contrast to Mailath and Postlewaite (1990), who show that when every agent has veto power, the probability of provision of the public good will be zero in *any* equilibrium of *any* mechanism in a large economy.

Appendix

Proof of Lemma 2:

In the proof of Lemma 1, we have shown that if all other agents use the $t(\eta_1, p)$-threshold strategy, then for agent i: (i) (yes, l) and (no, h) are suboptimal actions, and thus (yes, h)

[24] In addition, the ECSPR mechanism we propose has some attractive features that may make it potentially practical. First, the ECSPR mechanism is simple, in that once it is approved, it only asks agents to report one of two values (l or h). Second, the pivotal reward (and the punishment) used in the ECSPR mechanism is independent of the size of economy n, and thus is bounded even when the economy is large.

and (no, l) are the only possibilities; and (ii) (yes, h) is better than (no, l) if and only if v_i exceeds a threshold value. It can be verified that these two results hold as long as all other agents use symmetric consistent strategies. So, if a symmetric consistent-strategy profile forms a strict Bayesian Nash equilibrium, then it must be true that in the equilibrium, an agent chooses (yes, h) if and only if the agent's value is above a threshold and chooses (no, l) if and only if the agent's value is below the threshold. According to the proof of Lemma 1, such a threshold must be $t(\eta_1, p)$. $\square$

Proof of Lemma 3:

Suppose that $t(\eta_1, p) = \frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z(\eta_1,p)}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z(\eta_1,p)}{z(\eta_1,p)}+1-p}$ is increasing in η_1, then $z(\eta_1, p) = 1 - F(t(\eta_1, p))$ must be decreasing in η_1. This implies that $\frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z(\eta_1,p)}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z(\eta_1,p)}{z(\eta_1,p)}+1-p}$ is strictly decreasing in η_1 (in particular, note that (i) $\frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z}{z}+1-p} = \frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1}{z}-p\frac{n^*}{n-n^*}+1-p} = \frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1}{z}+1-p\frac{n}{n-n^*}} = \frac{C}{n} - \frac{\frac{n^*}{n-n^*}p\eta_1}{p\frac{n^*}{n-n^*}+(1-p\frac{n}{n-n^*})z}$ is increasing in z (since $1 - p\frac{n}{n-n^*} \geq 0$, which is because $p \leq \frac{n-n^*}{n}$), and (ii) $\frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p}{z}\eta_1}{p\frac{n^*}{n-n^*}\frac{1-z}{z}+1-p}$ is strictly decreasing in η_1). This contradicts the assumption that $t(\eta_1, p)$ is increasing in η_1. So, it must be the case that $t(\eta_1, p)$ is strictly decreasing in η_1.

Similarly, we can show that $t(\eta_1, p)$ is strictly decreasing in p. $\square$

Proof of Theorem 1:

If the pivotal reward $\eta_1 \in (c - F^{-1}(1-\alpha), c - \underline{v})$, then $t(\eta_1) \in (\underline{v}, F^{-1}(1-\alpha))$, and thus $z(\eta_1) \in (\alpha, 1)$. The probability that the public good is provided is equal to $p^* \times Prob(x = n^*) + Prob(x \geq n^*+1) \geq Prob(x \geq n^*+1)$ (where x is the number of agents who vote for the mechanism and report h in the $t(\eta_1)$-threshold strategy equilibrium). Note that x follows a binomial distribution with expected value $nz(\eta_1)$ and variance $nz(\eta_1)(1-z(\eta_1))$. Since $\frac{n^*+1}{n}$ approaches α as $n \to \infty$ and $z(\eta_1) > \alpha$, there exists an $N > 0$ such that $z(\eta_1) - \frac{n^*+1}{n} > 0$ for any $n \geq N$. Thus, for $n \geq N$, we have $Prob(x \geq n^* + 1) = Prob(\frac{x}{n} \geq \frac{n^*+1}{n}) = Prob(\frac{x}{n} - z(\eta_1) \geq \frac{n^*+1}{n} - z(\eta_1)) = 1 - Prob(\frac{x}{n} - z(\eta_1) < \frac{n^*+1}{n} - z(\eta_1)) \geq 1 - Prob(|\frac{x}{n} - z(\eta_1)| > z(\eta_1) - \frac{n^*+1}{n}) \geq 1 - \frac{Var(\frac{x}{n})}{(z(\eta_1)-\frac{n^*+1}{n})^2} = 1 - \frac{\frac{1}{n}z(\eta_1)(1-z(\eta_1))}{(z(\eta_1)-\frac{n^*+1}{n})^2}$, where the second-to-last inequality follows from the fact that $z(\eta_1) - \frac{n^*+1}{n} > 0$ for any $n \geq N$, and the last inequality follows from Chebyshev's inequality. This implies that $Prob(x \geq n^* + 1)$ approaches 1 as $n \to \infty$, which implies that the provision probability of the public good approaches 1 as $n \to \infty$. $\square$

Proof of Corollary 1:

Corollary 1 (ii) is a result of Theorem 1. We next prove Corollary 1 (i).

We call the game considered by Mailth and Postlewaite (1990) the *non-voting game.*[25] We will show that for any mechanism (say, $\mathcal{M}_1$) that satisfies ex post budget balance and any BNE (say, $\mathcal{E}_1$) of our *voting game* (with $\alpha = 1$) where the proposed mechanism is $\mathcal{M}_1$, we can find a (direct) mechanism $\mathcal{M}_2$ such that (i) $\mathcal{M}_2$ satisfies incentive compatibility, interim individual rationality, and ex post budget balance; and (ii) for any given realized valuations of agents, the probability of provision of the public good in the truthful equilibrium of the *non-voting game* where the proposed mechanism is $\mathcal{M}_2$ is the same as the probability of provision in the equilibrium $\mathcal{E}_1$ of the *voting game* (with $\alpha = 1$) where the proposed mechanism is $\mathcal{M}_1$.

Let $\mathcal{E}_1 = \{D_i^*(\cdot), r_i^*(\cdot)\}_{i=1}^n$ be a BNE of the *voting game* (with $\alpha = 1$) where the proposed mechanism is $\mathcal{M}_1 = \{q, \{t_i\}_{i=1}^n\}$, where $D_i^* : V_i = [\underline{v}, \overline{v}] \to \{\text{yes}, \text{no}\}$ is agent i's voting decision function (where "yes" refers to voting for the mechanism and "no" refers to voting against the mechanism) and $r_i^* : V_i \to V_i$ is agent i's valuation reporting function. Let $A_i = \{v_i \in V_i | D_i^*(v_i) = \text{yes}\}$ be the set of agent i's valuations at which agent i's equilibrium voting decision is "yes" and $R_i = \{v_i \in V_i | D_i^*(v_i) = \text{no}\}$ be the set of agent i's valuations at which agent i's equilibrium voting decision is "no." Let $v = (v_1, ..., v_n)$. We can define a new mechanism $\mathcal{M}_2 = \{\tilde{q}, \{\tilde{t}_i\}_{i=1}^n\}$ as follows: $\tilde{q}(v) = q(r_1^*(v), ..., r_n^*(v))$ and $\tilde{t}_i(v) = t_i(r_1^*(v), ..., r_n^*(v))$ (for all i) if v is such that $v_k \in A_k$ for all $k \in \{1, ..., n\}$, and $\tilde{q}(v) = 0$ and $\tilde{t}_i(v) = 0$ (for all i) otherwise.

It can be verified that in the *non-voting game* where the proposed mechanism is $\mathcal{M}_2$, all

[25] In the non-voting game, the principal proposes a public good provision mechanism $\{q, \{t_i\}_{i=1}^n\}$, and then each agent reports his valuation and the mechanism is implemented based on the reported valuations of all agents.

agents reporting truthfully is a BNE, i.e., $\mathcal{M}_2$ satisfies the *incentive compatibility* constraint (the proof is similar to the idea of the revelation principle and is omitted). In addition, note that the *interim individual rationality* constraint is satisfied by $\mathcal{M}_2$, because (i) if $v_i \in A_i$, then agent i's interim utility $\tilde{U}_i(v_i) = E_{v_{-i}}[v_i\tilde{q}_i(v) - \tilde{t}_i(v)] = E_{v_{-i}}[v_i\tilde{q}_i(v) - \tilde{t}_i(v)|v_{-i} \in A_{-i}] \times Prob(v_{-i} \in A_{-i}) + E_{v_{-i}}[v_i\tilde{q}_i(v) - \tilde{t}_i(v)|v_{-i} \notin A_{-i}] \times Prob(v_{-i} \notin A_{-i}) = E_{v_{-i}}[v_i\tilde{q}_i(v) - \tilde{t}_i(v)|v_{-i} \in A_{-i}] \times Prob(v_{-i} \in A_{-i}) = E_{v_{-i}}[v_i q_i(r_i^*(v_i), r_{-i}^*(v_{-i})) - t_i(r_i^*(v_i), r_{-i}^*(v_{-i}))|v_{-i} \in A_{-i}] \times Prob(v_{-i} \in A_{-i}) = U_i(v_i) \geq 0$[26], where $U_i(v_i)$ is agent i's interim utility in the equilibrium $\mathcal{E}_1$ of the voting game where the proposed mechanism is $\mathcal{M}_1$ and the inequality $U_i(v_i) \geq 0$ is due to the fact that $v_i \in A_i$, which implies that agent i is weakly worse off if he votes against the mechanism $\mathcal{M}_1$ (and obtains a zero payoff); and (ii) if $v_i \in R_i$, then $U_i(v_i) = 0$ according to the definition of $\mathcal{M}_2$. Finally, $\mathcal{M}_2$ satisfies *ex post budget balance* because $\mathcal{M}_1$ satisfies ex post budget balance.

It is easy to see that the probability of provision of the public good in the truthful equilibrium of the non-voting game where the proposed mechanism is $\mathcal{M}_2$ is the same as the probability of provision in the equilibrium $\mathcal{E}_1$ of the voting game (with $\alpha = 1$) where the proposed mechanism is $\mathcal{M}_1$. Therefore, Corollary 1 (i) holds because of the impossibility result of Mailath and Postelwaite (1990) (see their Theorems 1 and 2). □

Sketch of Proof of Lemma 4:

(i) The proof for this case consists of three steps.

First, it can be shown that in any strict BNE of the voting game, an agent's equilibrium action cannot be (yes, l), because it is a weakly dominated action.[27] In particular, for agent i, if $v_i \geq \frac{C}{n}$, then (yes, l) is weakly dominated by (yes, h). If $v_i < \frac{C}{n}$, then (yes, l) is

[26]Note that $A_{-i} = A_1 \times ... \times A_{i-1} \times A_{i+1} \times ... \times A_n$ (in addition, v_{-i} and r_{-i}^* are defined in a similar way).

[27]Note that in any strict BNE, a weakly dominated action cannot be an agent's equilibrium action (for almost any type of the agent). This is because in a strict BNE, if an agent type chooses a weakly dominated action, then the agent must be indifferent between this action and some other action (if not, then this weakly dominated action must be strictly worse than some other action, but this will contradict the fact that the agent chooses this weakly dominated action in equilibrium). However, a strict BNE requires that an agent's best response be unique (for almost any type of the agent). So, it must be true that an agent's equilibrium action in any strict BNE cannot be a weakly dominated action (for almost any type of the agent).

weakly dominated by (no, l). We next prove the former statement, and the analysis of the case of $v_i < \frac{C}{n}$ is similar and is omitted. In particular, suppose $v_i \geq \frac{C}{n}$. If the number of yes votes (excluding agent i) is $n^* - 2$ or fewer, then choosing (yes, l) and (yes, h) are indifferent, because the mechanism cannot be approved in any case. If the number of yes votes (excluding agent i) is equal to or greater than $n^* - 1$, then both (yes, h) and (yes, l) will induce the mechanism to be approved. In this case, we have the following four subcases. (a) If the number of h reports (excluding i) is equal to $n^* - 2$, then choosing (yes, h) will yield a positive payoff (in particular, η_2), while choosing (yes, l) will yield a zero payoff. (b) If the number of h reports (excluding i) is equal to $n^* - 1$, then choosing (yes, h) will yield a positive payoff (in particular, $p(v_i - (\frac{C}{n} - \eta_1))$), while choosing (yes, l) will yield a negative payoff $-\frac{n^*-1}{n-n^*+1}\eta_2$. (c) If the number of h reports (excluding i) is equal to n^*, then choosing (yes, h) will yield a payoff of $v_i - \frac{C}{n}$, while choosing (yes, l) will yield a smaller payoff $p(v_i - (\frac{C}{n} + \frac{n^*}{n-n^*}\eta_1))$ (noting that $v_i \geq \frac{C}{n}$). (d) If the number of h reports (excluding i) is equal to or greater than $n^* + 1$, then choosing (yes, h) and (yes, l) are indifferent.

So, we have shown that it is never optimal for an agent to choose (yes, l), regardless of the agent's valuation. This also implies that in any strict BNE, if an agent votes for the mechanism, he must report h.

Second, it can be shown that if $v_i \leq \frac{C}{n} - \eta_1$, then it is never optimal for agent i to choose (no, h) because it is weakly dominated by (no, l). In particular, these two actions can possibly make a difference for agent i only when all other agents' votes have at least n^* yes votes (which implies that $x_{-i} \geq n^*$ according to the result in the first step, where x_{-i} is the number of agents (excluding agent i) whose reported values are h). We have two subcases. (a) If all agents other than agent i have at least n^* yes votes and $x_{-i} \geq n^*+1$, then choosing (no, h) and choosing (no, l) are indifferent, because in either case, the public good will be provided and agent i will make the payment $\frac{C}{n}$. (b) If all agents other than agent i have at least n^* yes votes and $x_{-i} = n^*$, then agent i's utility will be $v_i - \frac{C}{n}$ if his action is (no, h), and his utility will be $p(v_i - (\frac{C}{n} + \frac{n^*}{n-n^*}\eta_1))$ if his action is (no, l). Using the fact that

$p = \frac{n-n^*}{n}$ and the fact that $v_i \leq \frac{C}{n} - \eta_1$, it can be verified that $v_i - \frac{C}{n} \leq p(v_i - (\frac{C}{n} + \frac{n^*}{n-n^*}\eta_1))$.

Third, it can be shown that if $v_i \leq \frac{C}{n} - \eta_1$, then it is never optimal for agent i to choose (yes, h) because it is weakly dominated by (no, h). These two actions make a difference only when agent i's vote is pivotal, i.e., when other agents' votes have exactly $n^* - 1$ yes votes (which implies that $x_{-i} \geq n^* - 1$, according to the result in the first step). We have two subcases. (a) If all agents' votes (excluding agent i) have exactly $n^* - 1$ yes votes and $x_{-i} = n^* - 1$, then choosing (yes, h) will yield a payoff of $p(v_i - (\frac{C}{n} - \eta_1))$, which is worse than choosing (no, h), which yields a payoff of zero. (b) If all agents' votes (excluding agent i) have exactly $n^* - 1$ yes votes and $x_{-i} \geq n^*$, then choosing (yes, h) will yield a payoff of $v_i - \frac{C}{n}$, which is also worse than choosing (no, h), which yields a payoff of zero.

The above three steps illustrate that in any strict BNE, agent i will choose (no, l) if $v_i \leq \frac{C}{n} - \eta_1$.

(ii) It can be shown that if $v_i > \frac{C}{n} - \eta_1$, then (no, l) is weakly dominated by (no, h) for agent i. The proof is similar to the second step in (i) and is omitted.

It can also be shown that if $v_i \geq \frac{C}{n}$, then (no, h) is weakly dominated by (yes, h) for agent i. The proof is similar to the third step in (i) and is omitted.

The above two steps, together with the first step in (i), imply that if $v_i \geq \frac{C}{n}$, then agent i's optimal action must be (yes, h).

(iii) The first step in (i) and the first step in (ii) imply that if $\frac{C}{n} - \eta_1 < v_i < \frac{C}{n}$, then (yes, l) and (no, l) cannot be agent i's equilibrium action. So, agent i will choose either (yes, h) or (no, h) (for a given v_i with $\frac{C}{n} - \eta_1 < v_i < \frac{C}{n}$, whether choosing (yes, h) is optimal or choosing (no, h) is optimal may depend on the strategies used by other agents). This implies that in a symmetric inconsistent-strategy strict BNE, there must exist a nonzero measure set $\hat{V} \subset (\frac{C}{n} - \eta_1, \frac{C}{n})$ such that if $v_i \in \hat{V}$, then agent i will choose (no, h), and if $v_i \in (\frac{C}{n} - \eta_1, \frac{C}{n}) \backslash \hat{V}$, then agent i will choose (yes, h) (if $\hat{V}$ has a zero measure, then the equilibrium will become a consistent-strategy BNE). $\square$

Proof of Lemma 5:

Similar to the analysis in Section 2.4, we have $t(\eta_1|m) = \frac{C}{n} - \frac{\frac{n^*}{n-n^*}\frac{p^*}{z(\eta_1|m)}\eta_1}{p^*\frac{n^*}{n-n^*}\frac{1-z(\eta_1|m)}{z(\eta_1|m)}+1-p^*}$ where $z(\eta_1|m) = 1 - F_m(t(\eta_1|m))$, where F_m is the distribution function of r_i when the common value component is m. Using the fact that $p^* = \frac{n-n^*}{n}$, it can be verified that $t(\eta_1|m) = c - \eta_1$ and $z(\eta_1|m) = 1 - F_m(t(\eta_1|m)) = 1 - F_m(c - \eta_1)$. Notice that $F_m(r) = Prob(r_i \leq r) = Prob(v_i + m \leq r) = Prob(v_i \leq r - m) = F(r - m)$. So, $z(\eta_1|m) = 1 - F(c - \eta_1 - m)$. Obviously, $z(\eta_1|m)$ is increasing in m. In addition, since F is strictly increasing on $[\underline{v}, \overline{v}]$ (as the density function of v_i is strictly positive on $[\underline{v}, \overline{v}]$), it must be true that $z(\eta_1|m)$ is strictly increasing in m at any m where $z(\eta_1|m) \in (0, 1)$.[28] $\square$

Proof of Theorem 2:

If $p = p^*$, then all agents using the $t(\eta_1|m)$-threshold strategy is a strict BNE (this is true even if $\eta_1 \leq 0$). The proof is similar to the proof of Lemma 1 and is omitted.

Notice that in equilibrium, the probability that the public good is provided is equal to $p^* \times Prob(x = n^*) + Prob(x \geq n^* + 1)$, where x is the number of agents who vote for the mechanism and report h in the $t(\eta_1^*|m)$-threshold strategy equilibrium. Note that x follows a binomial distribution $B(n, z(\eta_1^*|m))$. As n goes to infinity, $Prob(x = n^*)$ goes to zero. So, the probability that the public good is provided is $Prob(x \geq n^* + 1) = Prob(\frac{x}{n} \geq \frac{n^*+1}{n})$ in a large economy.

Let $m > m^*$. Using Lemma 5, we have $z(\eta_1^*|m) > z(\eta_1^*|m^*) = \alpha$. Since $\frac{n^*+1}{n}$ approaches α as $n \to \infty$, there exists an $N_1 > 0$ such that $z(\eta_1^*|m) - \frac{n^*+1}{n} > 0$ for any $n \geq N_1$. Thus, for $n \geq N_1$, we have $Prob(\frac{x}{n} \geq \frac{n^*+1}{n}) = Prob(\frac{x}{n} - z(\eta_1^*|m) \geq \frac{n^*+1}{n} - z(\eta_1^*|m)) = 1 - Prob(\frac{x}{n} - z(\eta_1^*|m) < \frac{n^*+1}{n} - z(\eta_1^*|m)) \geq 1 - Prob(|\frac{x}{n} - z(\eta_1^*|m)| > z(\eta_1^*|m) - \frac{n^*+1}{n}) \geq 1 - \frac{Var(\frac{x}{n})}{(z(\eta_1^*|m) - \frac{n^*+1}{n})^2} = 1 - \frac{\frac{1}{n}z(\eta_1^*|m)(1-z(\eta_1^*|m))}{(z(\eta_1^*|m) - \frac{n^*+1}{n})^2}$, where the second-to-last inequality follows from the fact that $z(\eta_1^*|m) - \frac{n^*+1}{n} > 0$ for any $n \geq N_1$ and the last inequality follows from Chebyshev's inequality. This implies that $Prob(\frac{x}{n} \geq \frac{n^*+1}{n})$ approaches 1 as n goes to infinity.

Let $m < m^*$. Using Lemma 5, we have $z(\eta_1^*|m) < z(\eta_1^*|m^*) = \alpha$. Since $\frac{n^*+1}{n}$ approaches

[28] Note that $z(\eta_1|m)$ is not strictly increasing in m at m where $z(\eta_1|m) = 0$ or 1, because $0 \leq z(\eta_1|m) \leq 1$ (in particular, the fact that $0 \leq z(\eta_1|m) \leq 1$ and the fact that $z(\eta_1|m)$ is increasing in m imply that $z(\eta_1|m)$ is a constant function in m at m where $z(\eta_1|m) = 0$ or at m where $z(\eta_1|m) = 1$).

α as $n \to \infty$, there exists an $N_2 > 0$ such that $\frac{n^*+1}{n} - z(\eta_1^*|m) > 0$ for any $n \geq N_2$. Thus, for any $n \geq N_2$, we have $Prob(\frac{x}{n} \geq \frac{n^*+1}{n}) = Prob(\frac{x}{n} - z(\eta_1^*|m) \geq \frac{n^*+1}{n} - z(\eta_1^*|m)) \leq \frac{Var(\frac{x}{n})}{(\frac{n^*+1}{n} - z(\eta_1^*|m))^2} = \frac{\frac{1}{n} z(\eta_1^*|m)(1-z(\eta_1^*|m))}{(\frac{n^*+1}{n} - z(\eta_1^*|m))^2}$, where the inequality follows from Chebyshev's inequality. This implies that $Prob(\frac{x}{n} \geq \frac{n^*+1}{n})$ approaches 0 as n goes to infinity. $\square$

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