Impact of Second-Order Uncertainty on the Efficiency of the 0.5-Double Auction

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Abstract This paper studies the impact of second-order uncertainty on the (ex-post) efficiency of the 0.5-double auction. We consider a discrete double auction model in which the seller's valuation is either $\underline{v}_s$ or $\overline{v}_s$ and the buyer's valuation is either $\underline{v}_b$ or $\overline{v}_b$ with $0 = \underline{v}_s < \underline{v}_b < \overline{v}_s < \overline{v}_b = 1$. The buyer's valuation and the seller's valuation are private information. We further assume that the buyer's *belief* about the seller's valuation is also private information (i.e., there is second-order uncertainty). We show that if $\overline{v}_b - \overline{v}_s$ and $\underline{v}_b - \underline{v}_s$ are sufficiently small, then the efficiency of the 0.5-double auction model with second-order uncertainty is *higher* than the efficiency of the 0.5-double auction model with first-order uncertainty. Our result thus serves as an example in which the introduction of higher-order uncertainty results in reducing, instead of increasing, the inefficiency of a bargaining model.

Keywords: Double auctions; Second-order uncertainty; Ex-post efficiency.

JEL classification: C72, C78

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1 Introduction

0.5-Double auction is a trading mechanism that is used to determine the terms of trade when a buyer and a seller bargain over an object. In the 0.5-double auction, the buyer submits a bid b and the seller submits a simultaneous offer s. Trade occurs if and only if b is greater than s. The buyer pays the seller $0.5b + 0.5s$ when trade occurs. If both traders' valuations are common knowledge, then the 0.5-double auction mechanism is ex-post efficient. If instead, each trader's valuation is unknown to the other trader, then the 0.5-double auction mechanism usually cannot attain full ex-post efficiency (e.g., Chatterjee and Samuelson (1983); Myerson and Satterthwaite (1983)).

In the literature about the double auction with incomplete information, it is usually assumed that each trader's belief about the other trader's valuation is common knowledge, i.e., there is only *first-order* uncertainty. This paper considers a discrete 0.5-double auction model[1] with *second-order* uncertainty and studies the impact of the introduction of second-order uncertainty on the efficiency of the 0.5-double auction model. More particularly, we assume that the seller's valuation $v_s \in \{\underline{v}_s, \overline{v}_s\}$ and the buyer's valuation $v_b \in \{\underline{v}_b, \overline{v}_b\}$ are both private information, where $0 = \underline{v}_s < \underline{v}_b < \overline{v}_s < \overline{v}_b = 1$. We depart from the literature by assuming that the buyer's belief about the seller's valuation is the buyer's private information.[2] We find that if $\overline{v}_b - \overline{v}_s$ and $\underline{v}_b - \underline{v}_s$ are sufficiently small, then the probability that trade occurs in the 0.5-double auction model with second-order uncertainty is higher than that in the 0.5-double auction model with first-order uncertainty. That is, the introduction of second-order uncertainty *reduces* the inefficiency of the model.

The intuition of our result is as follows. In both the 0.5-double auction model with first-

[1] In the discrete double auction model, the prior distribution of each trader's valuation is discrete. The reason why we consider the discrete double auction model is that in the discrete double auction model, the equilibrium of the model is unique (when the parameters of the model satisfy certain reasonable conditions). On the other hand, in the double auction model where the traders' valuations are continuously distributed, there are usually multiple equilibria in the model (see, e.g., Satterthwaite and Williams (1989); Leininger et al. (1989)).

[2] For the sake of simplicity, we still assume that the seller's belief about the buyer's valuation is common knowledge.

order uncertainty and the 0.5-double auction model with second-order uncertainty, there exists a unique regular Bayesian Nash equilibrium when $\overline{v}_b - \overline{v}_s$ and $\underline{v}_b - \underline{v}_s$ are sufficiently small. Moreover, in the unique equilibrium of either model, both the seller who has a high valuation and the buyer who has a low valuation report truthfully, and both the seller who has a low valuation and the buyer who has a high valuation use mixed strategies. In the model with first-order uncertainty, the seller with a low valuation uses a mixed strategy in such a way that the buyer with a high valuation is indifferent between reporting truthfully and misreporting. In the model with second-order uncertainty, the buyer with a high valuation may have different beliefs regarding the distribution of the seller's valuation. In the unique equilibrium, the seller who has a low valuation uses a mixed strategy in such a way that the buyer who has a high valuation and also puts a low probability on the low-valuation seller[3] is indifferent between reporting truthfully and misreporting. Thus, in the model with second-order uncertainty, the seller who has a low valuation is more likely to report truthfully than is the case for the model with first-order uncertainty.[4] A similar argument shows that the buyer who has a high valuation is more likely to report truthfully than is the case for the model with first-order uncertainty. Thus, the probability that trade occurs in the model with second-order uncertainty is greater than that in the model with first-order uncertainty.

There is a great deal of literature that has investigated the impact of higher-order uncertainty in various models. Weistein and Yildiz (2007) proposed a global stability condition, under which the maximum impact of higher-order uncertainties diminishes at an exponential rate as the order of uncertainty increases. Rubinstein (1989) studied an electronic mail game with "almost common knowledge" and found that the relaxation of the common knowledge assumption has a great impact on the strategic behavior of the players. Feinberg and

[3]We assume that there are two types of high-valuation buyer: the optimistic type (denoted by $\underline{t}$) and the pessimistic type (denoted by $\overline{t}$). The optimistic type puts a high probability over the low-valuation seller, and the pessimistic type puts a low probability over the low-valuation seller. That is, $p(\underline{v}_s|\underline{t}) > p(\underline{v}_s|\overline{t})$.

[4]This is because the buyer who has a high valuation and puts a low probability $p(\underline{v}_s|\overline{t})$ over the low-valuation seller has less incentive to misreport than the buyer who has a high valuation and puts a higher-than-$p(\underline{v}_s|\overline{t})$ probability over the low-valuation seller (in the model with first-order uncertainty, the probability that the high-valuation buyer puts over the low-valuation seller is $p(\underline{v}_s|\underline{t})P(\underline{t}) + p(\underline{v}_s|\overline{t})P(\overline{t})$, which is greater than $p(\underline{v}_s|\overline{t})$).

Skrzypacz (2005) studied the seller-offer model and found that delays in bargaining might arise when second-order uncertainty is introduced. Our paper focuses on the double auction model. Our model is interesting because it shows that the introduction of higher-order uncertainty results in reducing, instead of increasing, the inefficiency of a bargaining model. In the bargaining literature, the introduction of uncertainty usually results in the increase of inefficiency of the model (see, for example, Myerson and Satterthwaittee (1983) and Chatterjee and Samuelson (1983) for the introduction of first-order uncertainty, and Feinberg and Skrzypacz (2005) for the introduction of second-order uncertainty).[5]

This paper is organized as follows. Section 2 presents the 0.5-double auction model with second-order uncertainty. Section 3 analyzes the equilibrium of the 0.5-double auction model with second-order uncertainty and compares its efficiency with the efficiency of the 0.5-double auction model with first-order uncertainty. Concluding remarks are offered in Section 4.

2 0.5-Double Auction Model with Second-Order Uncertainty

There is one seller and one buyer who bargain over an indivisible object. The rule of the 0.5-double auction is as follows. The buyer submits a bid b and the seller submits an offer s. If b is greater than s, then trade occurs and the trading price is $0.5b + 0.5s$.

Let $v_s \in V_s$ denote the seller's valuation of the object and $v_b \in V_b$ denote the buyer's valuation of the object. v_s is the seller's private information and v_b is the buyer's private information. Assume that $V_s = \{\underline{v}_s, \overline{v}_s\}$, $V_b = \{\underline{v}_b, \overline{v}_b\}$ and $0 = \underline{v}_s < \underline{v}_b < \overline{v}_s < \overline{v}_b = 1$.[6]

[5] In contract to the bargaining literature, the introduction of uncertainty might induce *nicer* results in other literature. For example, Morris and Shin (1998) showed that the introduction of uncertainty helps select a unique equilibrium in the currency attack model (I am indebted to an anonymous referee for pointing out this result in the literature).

[6] It is a natural assumption to require the low-type seller to have a lower valuation than the low-type buyer, and to require the high-type seller to have a lower valuation than the high-type buyer. If, instead, $\underline{v}_b < \underline{v}_s < \overline{v}_b < \overline{v}_s$, then the dominant strategy for the buyer/seller is to report truthfully, and ex-post efficiency can thus always be achieved (see also Myerson and Satterthwaite (1983) for a specific example).

Denote $P(v_b = \overline{v}_b) = \overline{p}$ and $P(v_s = \underline{v}_s) = \underline{q}$. Assume that $0 < \overline{p} < 1$ and $0 < \underline{q} < 1$.

Assume that the buyer's (first-order) belief about the seller's valuation is the buyer's private information.[7] We use t to denote this private information. Each realized t corresponds to a distribution over V_s, denoted by $p(v_s|t)$. Assume that t can take only two values, $\underline{t}$ and $\overline{t}$, where $p(v_s = \underline{v}_s|\underline{t}) > p(v_s = \underline{v}_s|\overline{t})$. Let $T = \{\underline{t}, \overline{t}\}$. Assume that players have a common prior about t. Therefore, both the $\underline{v}_s$-type seller's belief and the $\overline{v}_s$-type seller's belief about t are common knowledge.[8] That is, there is only second-order uncertainty and no higher-order uncertainty. Finally, we assume that v_b is independent of v_s and t.

Notice that if $\underline{t} = \overline{t}$, then the model degenerates to the 0.5-double auction model with first-order uncertainty.

V_S is the seller's type space and $V_b \times T$ is the buyer's type space. A (mixed) strategy of seller, $s(\cdot)$, is a mapping from V_S to $\Delta(V_s)$. Similarly, a (mixed) strategy of buyer, $b(\cdot)$, is a mapping from $V_b \times T$ to $\Delta(V_b)$.

The equilibrium concept we use is the Bayesian Nash equilibrium (henceforth BNE). Notice that when $\overline{v}_b - \overline{v}_s$ and $\underline{v}_b - \underline{v}_s$ are sufficiently small, it is always a BNE that the seller reports $\overline{v}_s$ regardless of the seller type and the buyer reports truthfully (using Lemma 1). Similarly, it is always a BNE that the buyer reports $\underline{v}_b$ regardless of the buyer type and the seller reports truthfully.[9] In order to rule out the above two uninteresting cases,[10] we need the following equilibrium refinement.[11]

[7]For the sake of simplicity, we assume throughout the paper that the seller's belief about the buyer's valuation is common knowledge. This "one-sided" second-order uncertainty setting also appears in Feinberg and Skrzypacz (2005).

[8]The seller's belief about t is determined by $p(t|v_s) = \frac{p(v_s|t)p(t)}{p(v_s)}$, where $p(t)$ is the common prior about t.

[9]More particularly, the sufficient condition for the first equilibrium is that $\underline{m} < \overline{p}$ and the sufficient condition for the second equilibrium is that $\overline{m} < P(\underline{v}_s|\overline{t})$, where $\overline{m}$ and $\underline{m}$ (defined in the next section) are sufficiently small if $\overline{v}_b - \overline{v}_s$ and $\underline{v}_b - \underline{v}_s$ are sufficiently small (see Appendix 3).

[10]Notice that in the first equilibrium, the probability that trade should occur, but does not occur, always equals $P(\underline{v}_s)P(\underline{v}_b)$, regardless of whether or not there is second-order uncertainty. In the second equilibrium, the probability that trade should occur, but does not occur, always equals $P(\overline{v}_s)P(\overline{v}_b)$, regardless of whether or not there is second-order uncertainty.

[11]Noticing that in any BNE, the seller with a high valuation and the buyer with a low valuation will always choose to report truthfully, the equilibrium refinement below requires the seller's/buyer's strategy to be a function that is strictly "increasing" in his offer/bid, i.e., as the seller's/buyer's valuation changes from

Definition 1. *A regular Bayesian Nash equilibrium* (s, b) *is a Bayesian Nash equilibrium that satisfies* $p(s(v_s) = \underline{v}_s) > 0$ *and* $p(b(v_b, t) = \overline{v}_b) > 0$.

3 Main Result

3.1 Equilibrium Analysis

We will show that if $\overline{v}_b - \overline{v}_s$ and $\underline{v}_b - \underline{v}_s$ are sufficiently small, then there is a unique regular BNE in the 0.5-double auction model with second-order uncertainty.

First notice that in any BNE, the seller with a high valuation and the buyer with a low valuation will always choose to report truthfully. That is, if (s, b) is a BNE, then we must have $P(s(\overline{v}_s) = \overline{v}_s) = 1$ and $P(b(\underline{v}_b, t) = \underline{v}_b) = 1$ for any $t \in T$. The following lemma characterizes the equilibrium strategy of the seller with a low valuation and the equilibrium strategy of the buyer with a high valuation. Define $\overline{m} = \dfrac{\overline{v}_b - \overline{v}_s}{2\overline{v}_b - \underline{v}_b - \underline{v}_s}$ and $\underline{m} = \dfrac{\underline{v}_b - \underline{v}_s}{\underline{v}_b + \overline{v}_s - 2\underline{v}_s}$.

Lemma 1. *If* (s, b) *is a BNE, then we have:*

(i) ($\underline{v}_s$-type seller) if $P(b(v_b, t) = \overline{v}_b | \underline{v}_s) \substack{< \\ >} \underline{m}$, *then* $P(s(\underline{v}_s) = \underline{v}_s) = \substack{1 \\ 0}$; *if* $P(b(v_b, t) = \overline{v}_b | \underline{v}_s) = \underline{m}$, *then the* $\underline{v}_s$*-type seller is indifferent between reporting* $\underline{v}_s$ *and reporting* $\overline{v}_s$;

(ii) ($(\overline{v}_b, \overline{t})$-type buyer) if $P(s(v_s) = \underline{v}_s | (\overline{v}_b, \overline{t})) \substack{< \\ >} \overline{m}$, *then* $P(b(\overline{v}_b, \overline{t}) = \overline{v}_b) = \substack{1 \\ 0}$; *if* $P(s(v_s) = \underline{v}_s | (\overline{v}_b, \overline{t})) = \overline{m}$, *then the* $(\overline{v}_b, \overline{t})$*-type buyer is indifferent between reporting* $\overline{v}_b$ *and reporting* $\underline{v}_b$;

(iii) ($(\overline{v}_b, \underline{t})$-type buyer) if $P(s(v_s) = \underline{v}_s | (\overline{v}_b, \underline{t})) \substack{< \\ >} \overline{m}$, *then* $P(b(\overline{v}_b, \underline{t}) = \overline{v}_b) = \substack{1 \\ 0}$; *if* $P(s(v_s) = \underline{v}_s | (\overline{v}_b, \underline{t})) = \overline{m}$, *then the* $(\overline{v}_b, \underline{t})$*-type buyer is indifferent between reporting* $\overline{v}_b$ *and reporting* $\underline{v}_b$.

Proof: If the $\underline{v}_s$-type seller reports truthfully, then his expected payoff is $P(b(v_b, t) = \overline{v}_b | \underline{v}_s) \times (\frac{1}{2}(\overline{v}_b + \underline{v}_s) - \underline{v}_s) + P(b(v_b, t) = \underline{v}_b | \underline{v}_s) \times (\frac{1}{2}(\underline{v}_b + \underline{v}_s) - \underline{v}_s)$. If the $\underline{v}_s$-type seller reports

low valuation to high valuation, the probability that he reports a high valuation strictly increases. See also Chatterjee and Samuelson (1983) and Satterthwaite and Williams (1989) for a similar equilibrium refinement for the double auction model where traders' valuations are continuously distributed.

$\overline{v}_s$, then his expected payoff is $P(b(v_b,t)=\overline{v}_b|\underline{v}_s)\times(\frac{1}{2}(\overline{v}_b+\overline{v}_s)-\underline{v}_s)$.

It can be readily verified that $P(b(v_b,t)=\overline{v}_b|\underline{v}_s)\times(\frac{1}{2}(\overline{v}_b+\underline{v}_s)-\underline{v}_s)+P(b(v_{b,t})=\underline{v}_b|\underline{v}_s)\times(\frac{1}{2}(\underline{v}_b+\underline{v}_s)-\underline{v}_s)>P(b(v_b,t)=\overline{v}_b|\underline{v}_s)\times(\frac{1}{2}(\overline{v}_b+\overline{v}_s)-\underline{v}_s)$ if and only if $P(b(v_b,t)=\overline{v}_b|\underline{v}_s)<\underline{m}$. Thus, if $P(b(v_b,t)=\overline{v}_b|\underline{v}_s)<\underline{m}$, then $P(s(\underline{v}_s)=\underline{v}_s)=1$. If $P(b(v_b,t)=\overline{v}_b|\underline{v}_s)>\underline{m}$, then $P(s(\underline{v}_s)=\underline{v}_s)=0$. If $P(b(v_b,t)=\overline{v}_b|\underline{v}_s)=\underline{m}$, then the $\underline{v}_s$-type seller is indifferent between reporting $\underline{v}_s$ and reporting $\overline{v}_s$.

The proof for the buyer is similar and is omitted. □

Using the fact that $P(s(\overline{v}_s)=\overline{v}_s)=1$ and $P(b(\underline{v}_b,t)=\underline{v}_b)=1$ (for any $t\in T$) for any BNE and the fact that v_b is independent of v_s and t, we obtain the following lemma, which is useful for the purpose of simplifying the calculations of the conditions obtained in Lemma 1.

Lemma 2. *If (s,b) is a BNE, then we have:*

(i) $P(b(v_b,t)=\overline{v}_b|\underline{v}_s)=\overline{p}[P(b(\overline{v}_b,\overline{t})=\overline{v}_b)P(\overline{t}|\underline{v}_s)+P(b(\overline{v}_b,\underline{t})=\overline{v}_b)P(\underline{t}|\underline{v}_s)]$;

(ii) $P(s(v_s)=\underline{v}_s|(\overline{v}_b,\overline{t}))=P(s(\underline{v}_s)=\underline{v}_s)P(\underline{v}_s|\overline{t})$;

(iii) $P(s(v_s)=\underline{v}_s|(\overline{v}_b,\underline{t}))=P(s(\underline{v}_s)=\underline{v}_s)P(\underline{v}_s|\underline{t})$.

Proof: see Appendix 1. □

The following theorem shows that if $\underline{m}$ and $\overline{m}$ are sufficiently small, then there is a unique regular BNE in the model.

Theorem 1. *Assume that $\underline{m}<\overline{p}P(\overline{t}|\underline{v}_s)$ and $\overline{m}<P(\underline{v}_s|\overline{t})$, then the 0.5-double auction model with second-order uncertainty has a unique regular BNE, which is denoted by (s^*,b^*). Moreover, (s^*,b^*) is such that $P(s^*(\underline{v}_s)=\underline{v}_s)=\dfrac{\overline{m}}{P(\underline{v}_s|\overline{t})}$, $P(b^*(\overline{v}_b,\overline{t})=\overline{v}_b)=\dfrac{\underline{m}}{\overline{p}P(\overline{t}|\underline{v}_s)}$ and $P(b^*(\overline{v}_b,\underline{t})=\overline{v}_b)=0$.*

Proof: Let (s^*,b^*) be a regular BNE. We will first show that $P(b^*(\overline{v}_b,\overline{t})=\overline{v}_b)\in(0,1)$ and $P(b^*(\overline{v}_b,\underline{t})=\overline{v}_b)=0$. In order to show this, we will show that the following three cases cannot be a regular BNE.

(i) $P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b) = 1$ and $P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b) \in [0, 1]$.

In this case, we have $P(b^*(v_b, t) = \overline{v}_b | \underline{v}_s) = \overline{p}[P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b)P(\overline{t}|\underline{v}_s) + P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b)P(\underline{t}|\underline{v}_s)] \geq \overline{p}P(\overline{t}|\underline{v}_s) > \underline{m}$, where the first equality follows from Lemma 2 (i). Thus, $P(s^*(\underline{v}_s) = \underline{v}_s) = 0$ (Lemma 1 (i)). Since $P(s^*(\overline{v}_s) = \underline{v}_s) = 0$ in any BNE, we thus have $P(s^*(v_s) = \underline{v}_s) = 0$, which implies that (s^*, b^*) cannot be a *regular* BNE.

(ii) $P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b) \in [0, 1)$ and $P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b) \in (0, 1]$.

Since $P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b) \in [0, 1)$, we must have $P(s^*(v_s) = \underline{v}_s | (\overline{v}_b, \overline{t})) \geq \overline{m}$ (Lemma 1 (ii)). That is $P(s^*(\underline{v}_s) = \underline{v}_s)P(\underline{v}_s|\overline{t}) \geq \overline{m}$. (Lemma 2 (ii)).

Since $P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b) \in (0, 1]$, we must have $P(s^*(v_s) = \underline{v}_s | (\overline{v}_b, \underline{t})) \leq \overline{m}$ (Lemma 1 (iii)). That is $P(s^*(\underline{v}_s) = \underline{v}_s)P(\underline{v}_s|\underline{t}) \leq \overline{m}$ (Lemma 2 (iii)). Since $P(\underline{v}_s|\underline{t}) > P(\underline{v}_s|\overline{t})$, then $P(s^*(\underline{v}_s) = \underline{v}_s)P(\underline{v}_s|\overline{t}) < P(s^*(\underline{v}_s) = \underline{v}_s)P(\underline{v}_s|\underline{t}) \leq \overline{m}$, which is a contradiction with $P(s^*(\underline{v}_s) = \underline{v}_s)P(\underline{v}_s|\overline{t}) \geq \overline{m}$!

(iii) $P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b) = 0$ and $P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b) = 0$.

In this case, $P(b^*(v_b, t) = \overline{v}_b) = P(b^*(\underline{v}_b, t) = \overline{v}_b)P(\underline{v}_b) + P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b)P(\overline{v}_b, \overline{t}) + P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b)P(\overline{v}_b, \underline{t}) = 0$. Thus, (s^*, b^*) cannot be a *regular* BNE.

We thus have proved that if (s^*, b^*) is a regular BNE, then we must have $P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b) \in (0, 1)$ and $P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b) = 0$.

$P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b) \in (0, 1)$ implies that $P(s^*(v_s) = \underline{v}_s | (\overline{v}_b, \overline{t})) = \overline{m}$ (Lemma 1 (ii)). That is $P(s^*(\underline{v}_s) = \underline{v}_s)P(\underline{v}_s|\overline{t}) = \overline{m}$. So we have $P(s^*(\underline{v}_s) = \underline{v}_s) = \dfrac{\overline{m}}{P(\underline{v}_s|\overline{t})}$.

Since $P(s^*(\underline{v}_s) = \underline{v}_s) = \dfrac{\overline{m}}{P(\underline{v}_s|\overline{t})} \in (0, 1)$, we must have $P(b^*(v_b, t) = \overline{v}_b | \underline{v}_s) = \underline{m}$ (Lemma 1 (i)). So we have $P(b^*(v_b, t) = \overline{v}_b | \underline{v}_s) = \overline{p}[P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b)P(\overline{t}|\underline{v}_s) + P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b)P(\underline{t}|\underline{v}_s)] = \overline{p}P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b)P(\overline{t}|\underline{v}_s) = \underline{m}$, where the second equality follows from the fact that $P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b) = 0$. That is, $P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b) = \dfrac{\underline{m}}{\overline{p}P(\overline{t}|\underline{v}_s)}$. □

In Appendix 3, we show that a sufficient condition for $\overline{m}$ and $\underline{m}$ being sufficiently small is that $\overline{v}_b - \overline{v}_s$ and $\underline{v}_b - \underline{v}_s$ are sufficiently small.

A corollary of Theorem 1 is that the 0.5-double auction model with first-order uncertainty has a unique regular BNE when $\underline{m}$ and $\overline{m}$ are sufficiently small. Notice that in the case of first-order uncertainty, we have $\underline{t} = \overline{t}$. Thus, $P(\overline{t}|\underline{v}_s) = 1$ and $P(\underline{v}_s|\overline{t}) = P(\underline{v}_s)$.[12]

Corollary 1. *Assume that $\underline{m} < \overline{p}$ and $\overline{m} < \underline{q}$, then the 0.5-double auction model with first-order uncertainty has a unique regular BNE, which is denoted by $(\hat{s}, \hat{b})$. Moreover, $(\hat{s}, \hat{b})$ is such that $P(\hat{s}(\underline{v}_s) = \underline{v}_s) = \frac{\overline{m}}{\underline{q}}$ and $P(\hat{b}(\overline{v}_b) = \overline{v}_b) = \frac{\underline{m}}{\overline{p}}$.*

3.2 Comparison of Efficiency

The (ex-post) *inefficiency* under the strategy (s, b) is defined as the probability that trade should occur, but does not occur. That is, inefficiency $I = P(s(v_s) = \overline{v}_s, b(v_b, t) = \underline{v}_b, v_s = \underline{v}_s, v_b = \underline{v}_b) + P(s(v_s) = \overline{v}_s, b(v_b, t) = \underline{v}_b, v_s = \underline{v}_s, v_b = \overline{v}_b) + P(s(v_s) = \overline{v}_s, b(v_b, t) = \underline{v}_b, v_s = \overline{v}_s, v_b = \overline{v}_b)$.

This subsection will compare the inefficiency of the 0.5-double auction model with first-order uncertainty and the inefficiency of the 0.5-double auction model with second-order uncertainty for the case where $\underline{m}$ and $\overline{m}$ are small.[13] We find that when $\underline{m}$ and $\overline{m}$ are sufficiently small, the introduction of second-order uncertainty reduces the inefficiency of the model.

Let I_1 be the inefficiency of the 0.5-double auction model with first-order uncertainty (i.e., the inefficiency under $(\hat{s}, \hat{b})$) and let I_2 be the inefficiency of the 0.5-double auction model with second-order uncertainty (i.e., the inefficiency under (s^*, b^*)). We obtain the following result.

Theorem 2. *If $\underline{m} < \overline{p}P(\overline{t}|\underline{v}_s)$ and $\overline{m} < P(\underline{v}_s|\overline{t})$, then $I_2 < I_1$.*

[12] We require that the prior probability of v_s in the model with first-order uncertainty be the same as the prior probability of v_s in the model with second-order uncertainty. If they are not equal, then it is meaningless to compare the efficiency of the two models.

[13] Notice that when $\underline{m}$ and $\overline{m}$ are sufficiently small, both the 0.5-double auction model with first-order uncertainty and the 0.5-double auction model with second-order uncertainty have a *unique* regular BNE. Thus, the inefficiency of the 0.5-double auction model with first-order uncertainty refers to the inefficiency under $(\hat{s}, \hat{b})$ and the inefficiency of the 0.5-double auction model with second-order uncertainty refers to the inefficiency under (s^*, b^*).

Proof: $I_1 = I_1^1 + I_1^2 + I_1^3$ where $I_1^1 = p(\hat{s}(v_s) = \overline{v}_s, \hat{b}(v_b) = \underline{v}_b, v_s = \underline{v}_s, v_b = \underline{v}_b)$, $I_1^2 = p(\hat{s}(v_s) = \overline{v}_s, \hat{b}(v_b) = \underline{v}_b, v_s = \underline{v}_s, v_b = \overline{v}_b)$ and $I_1^3 = p(\hat{s}(v_s) = \overline{v}_s, \hat{b}(v_b) = \underline{v}_b, v_s = \overline{v}_s, v_b = \overline{v}_b)$.

Similarly, $I_2 = I_2^1 + I_2^2 + I_2^3$ where $I_2^1 = p(s^*(v_s) = \overline{v}_s, b^*(v_b, t) = \underline{v}_b, v_s = \underline{v}_s, v_b = \underline{v}_b)$, $I_2^2 = p(s^*(v_s) = \overline{v}_s, b^*(v_b, t) = \underline{v}_b, v_s = \underline{v}_s, v_b = \overline{v}_b)$ and $I_2^3 = p(s^*(v_s) = \overline{v}_s, b^*(v_b, t) = \underline{v}_b, v_s = \overline{v}_s, v_b = \overline{v}_b)$.

Now we have:

$$\begin{aligned} I_1^1 =& p(\hat{s}(v_s) = \overline{v}_s, \hat{b}(v_b) = \underline{v}_b, v_s = \underline{v}_s, v_b = \underline{v}_b) \\ =& P(\hat{s}(\underline{v}_s) = \overline{v}_s) P(\hat{b}(\underline{v}_b) = \underline{v}_b) P(v_s = \underline{v}_s) P(v_b = \underline{v}_b) \\ =& (1 - \frac{\overline{m}}{\underline{q}}) \underline{q} (1 - \overline{p}); \end{aligned}$$

$$\begin{aligned} I_2^1 =& P(s^*(v_s) = \overline{v}_s, b^*(v_b, t) = \underline{v}_b, v_s = \underline{v}_s, v_b = \underline{v}_b) \\ =& P(s^*(\underline{v}_s) = \overline{v}_s) P(b^*(\underline{v}_b, t) = \underline{v}_b | v_s = \underline{v}_s) P(v_s = \underline{v}_s) P(v_b = \underline{v}_b) \\ =& (1 - \frac{\overline{m}}{P(\underline{v}_s|\overline{t})}) \underline{q} (1 - \overline{p}); \end{aligned}$$

$$\begin{aligned} I_1^2 =& P(\hat{s}(v_s) = \overline{v}_s, \hat{b}(v_b) = \underline{v}_b, v_s = \underline{v}_s, v_b = \overline{v}_b) \\ =& P(\hat{s}(\underline{v}_s) = \overline{v}_s) P(\hat{b}(\overline{v}_b) = \underline{v}_b) P(v_s = \underline{v}_s) P(v_b = \overline{v}_b) \\ =& (1 - \frac{\overline{m}}{\underline{q}})(1 - \frac{\underline{m}}{\overline{p}}) \underline{q}\overline{p}; \end{aligned}$$

$$\begin{aligned} I_2^2 =& P(s^*(v_s) = \overline{v}_s, b^*(v_b, t) = \underline{v}_b, v_s = \underline{v}_s, v_b = \overline{v}_b) \\ =& P(s^*(\underline{v}_s) = \overline{v}_s) P(b^*(\overline{v}_b, t) = \underline{v}_b | v_s = \underline{v}_s) P(v_s = \underline{v}_s) P(v_b = \overline{v}_b) \\ =& P(s^*(\underline{v}_s) = \overline{v}_s)[P(b^*(\overline{v}_b, \overline{t}) = \underline{v}_b) P(\overline{t}|\underline{v}_s) + P(b^*(\overline{v}_b, \underline{t}) = \underline{v}_b) P(\underline{t}|\underline{v}_s)] \underline{q}\overline{p} \\ =& P(s^*(\underline{v}_s) = \overline{v}_s)[(1 - \frac{\underline{m}}{\overline{p} P(\overline{t}|\underline{v}_s)}) P(\overline{t}|\underline{v}_s) + P(\underline{t}|\underline{v}_s)] \underline{q}\overline{p} \\ =& (1 - \frac{\overline{m}}{P(\underline{v}_s|\overline{t})})(1 - \frac{\underline{m}}{\overline{p}}) \underline{q}\overline{p}; \end{aligned}$$

$$
\begin{aligned}
I_1^3 =& P(\hat{s}(v_s) = \overline{v}_s, \hat{b}(v_b) = \underline{v}_b, v_s = \overline{v}_s, v_b = \overline{v}_b) \\
=& P(\hat{s}(\overline{v}_s) = \overline{v}_s) P(\hat{b}(\overline{v}_b) = \underline{v}_b) P(v_s = \overline{v}_s) P(v_b = \overline{v}_b) \\
=& (1 - \frac{m}{\overline{p}})(1 - \underline{q})\overline{p};
\end{aligned}
$$

$$
\begin{aligned}
I_2^3 =& P(s^*(v_s) = \overline{v}_s, b^*(v_b, t) = \underline{v}_b, v_s = \overline{v}_s, v_b = \overline{v}_b) \\
=& P(s^*(\overline{v}_s) = \overline{v}_s) P(b^*(\overline{v}_b, t) = \underline{v}_b | v_s = \overline{v}_s) P(v_s = \overline{v}_s) P(v_b = \overline{v}_b) \\
=& P(s^*(\overline{v}_s) = \overline{v}_s)[P(b^*(\overline{v}_b, \overline{t}) = \underline{v}_b) P(\overline{t}|\overline{v}_s) + P(b^*(\overline{v}_b, \underline{t}) = \underline{v}_b) P(\underline{t}|\overline{v}_s)](1 - \underline{q})\overline{p} \\
=& [(1 - \frac{m}{\overline{p} P(\overline{t}|\underline{v}_s)}) P(\overline{t}|\overline{v}_s) + P(\underline{t}|\overline{v}_s)](1 - \underline{q})\overline{p} \\
=& (1 - \frac{m}{\overline{p}} \frac{P(\overline{t}|\overline{v}_s)}{P(\overline{t}|\underline{v}_s)})(1 - \underline{q})\overline{p}.
\end{aligned}
$$

Since $P(\underline{v}_s|\underline{t}) > P(\underline{v}_s|\overline{t})$, we have $\underline{q} = P(\underline{v}_s) = P(\underline{v}_s|\overline{t})P(\overline{t}) + P(\underline{v}_s|\underline{t})P(\underline{t}) > P(\underline{v}_s|\overline{t})$. So we have $I_2^1 < I_1^1$ and $I_2^2 < I_1^2$.

Using the fact that $P(\overline{t}|\overline{v}_s) > P(\overline{t}|\underline{v}_s)$ (see Appendix 2 for the proof), we have $I_2^3 < I_1^3$.

Thus we have proved $I_2 < I_1$. □

Remark: According to our definition, inefficiency is the sum of three probabilities. These three probabilities correspond to the three cases where trade should occur but does not occur. In these three cases, the loss from trade is $\underline{v}_b - \underline{v}_s$, $\overline{v}_b - \underline{v}_s$ and $\overline{v}_b - \overline{v}_s$ respectively. The proof above shows that for each of the three cases, the probability that trade should occur, but does not occur, is smaller in the model with second-order uncertainty than that in the model with first-order uncertainty. This implies that even if we define inefficiency as the *expected loss* from trade due to incomplete information, then we will still come to the same conclusion. That is, the inefficiency (the expected loss) in the model with second-order uncertainty is smaller than the inefficiency (the expected loss) in the model with first-order uncertainty.

Example: Let $\overline{v}_b = 1$, $\overline{v}_s = 0.9$, $\underline{v}_b = 0.1$, and $\underline{v}_s = 0$. Let $\underline{q} = P(v_s = \underline{v}_s) = 0.5$, $\overline{p} = P(v_b = \overline{v}_b) = 0.5$, $P(\underline{v}_s|\overline{t}) = 0.4$, and $P(\underline{v}_s|\underline{t}) = 0.6$ (these equalities imply that $P(\overline{t}|\underline{v}_s) = 0.4$ and $P(\overline{t}) = P(\underline{t}) = 0.5$). It can be readily verified that $\overline{m} = 0.0526$, which is less than $P(\underline{v}_s|\overline{t}) = 0.4$, and $\underline{m} = 0.1$, which is less than $\overline{p}P(\overline{t}|\underline{v}_s) = 0.2$. Thus the conditions of Theorem 1, 2 and Corollary 1 are satisfied. Then $P(s^*(\underline{v}_s) = \underline{v}_s) = \dfrac{\overline{m}}{P(\underline{v}_s|\overline{t})} = 0.1315$, $P(b^*(\overline{v}_b, \overline{t}) = \overline{v}_b) = \dfrac{\underline{m}}{\overline{p}P(\overline{t}|\underline{v}_s))} = 0.5$ (which implies that $P(b^*(\overline{v}_b, t) = \overline{v}_b) = 0.25$, given that $P(b^*(\overline{v}_b, \underline{t}) = \overline{v}_b) = 0$ and $P(\overline{t}) = P(\underline{t}) = 0.5$), $P(\hat{s}(\underline{v}_s) = \underline{v}_s) = \dfrac{\overline{m}}{\underline{q}} = 0.1052$, and $P(\hat{b}(\overline{v}_b) = \overline{v}_b) = \dfrac{\underline{m}}{\overline{p}} = 0.2$. Thus, both the low-valuation seller and the high-valuation buyer report true valuations with higher probabilities in the 0.5-double auction model with second-order uncertainty than is the case for the 0.5-double auction model with first-order uncertainty. Actually, it can be verified that the probability that trade should occur, but does not occur in the model with first-order uncertainty is 0.6026, and it is 0.5657 in the model with second-order uncertainty. Finally, it should be noted that in both models, the probability that trade should occur is the same and equals $P(v_b = \overline{v}_b) + P(v_b = \underline{v}_b, v_s = \underline{v}_s) = 0.75$.

4 Concluding Remarks

This paper studies the impact of second-order uncertainty on the (ex-post) efficiency of the discrete 0.5-double auction. We assume that the buyer's *belief* about the seller's valuation is private information. We show that if $\overline{m}$ and $\underline{m}$ are sufficiently small, then the efficiency of the 0.5-double auction model with second-order uncertainty is *higher* than that of the model with first-order uncertainty. Thus, the introduction of second-order uncertainty *reduces* the inefficiency of the discrete double auction model when the parameters of the model satisfy certain reasonable conditions. My result thus implies that the negative result in the double auction literature (i.e., inefficiency of trade arises when traders' valuations are private information) might be less severe than we originally imagined if the common knowledge assumption (about the first-order beliefs) is relaxed and higher-order uncertainty is introduced in the model.

My model (i.e., the model with second-order uncertainty) and the model with first-order uncertainty only differ in players' beliefs. My analysis shows that the pure fact that the buyer's first-order belief is private information in my model induces the seller with a low valuation to report truthfully with a higher probability than is the case for the model with first-order uncertainty. It also induces the buyer with a high valuation to report truthfully with a higher probability than is the case for the model with first-order uncertainty. As a result, trade occurs more often in my model than the model with first-order uncertainty.

In my model, the buyer essentially has three types: $(\overline{v}_b, \overline{t})$, $(\overline{v}_b, \underline{t})$, and $\underline{v}_b$.[14] However, my model is different from a 3-type model of valuations. In particular, the two models differ in the following ways. First, unlike the usual 3-type model of valuations where the buyer's type and the seller's type are independent, the buyer's type (v_b, t) in my model is *correlated* with the seller's type v_s because t and v_s are correlated.[15] Second, if we consider a 3-type model of valuations, then it is impossible to compare the efficiency of the model with that of a 2-type model of valuations (i.e., the model with first-order uncertainty considered in this paper), because the fundamentals (i.e., players' valuations) of the two models are different.[16]

When $\overline{m}$ and $\underline{m}$ become large, it can be verified that the regular BNE in the 0.5-double auction model with second-order uncertainty might not be unique. In this case, the introduction of second-order uncertainty has an ambiguous effect on the inefficiency of the 0.5-double auction model. For some equilibria, the inefficiency might be lower in the model with second-order uncertainty than that in the model with first-order uncertainty. But for some other equilibria, the inefficiency might be higher in the model with second-order uncertainty than that in the model with first-order uncertainty.

[14]Actually, there are two types of $\underline{v}_b$-type buyer: the $(\underline{v}_b, \overline{t})$-type buyer and the $(\underline{v}_b, \underline{t})$-type buyer. However, both the $(\underline{v}_b, \overline{t})$-type buyer and the $(\underline{v}_b, \underline{t})$-type buyer report $\underline{v}_b$ in any BNE, and thus they can be regarded as one type.

[15]However, my model is also different from a model with correlated valuations, because the seller's *valuation* and the buyer's *valuation* in my model are independent.

[16]Notice that my model and the 2-type model of valuations share the same fundamentals, and it is thus possible to compare the efficiency of those two models.

Finally, although this paper focuses on the 0.5-double auction model, it should be noted that the main result obtained in this paper can be easily extended to the k-double auction model where $0 < k < 1$.

Appendix 1 (proof of Lemma 2)

Assume that (s, b) is a BNE. We will prove (i). The proofs for (ii) and (iii) are similar and are thus omitted.

We have:

$$
\begin{aligned}
& P(b(v_b,t)=\overline{v}_b|\underline{v}_s) \\
=& \frac{P(b(v_b,t)=\overline{v}_b, v_s=\underline{v}_s)}{P(v_s=\underline{v}_s)} \\
=& \frac{P(b(v_b,t)=\overline{v}_b, v_s=\underline{v}_s)}{P(v_b=\overline{v}_b, v_s=\underline{v}_s)} P(v_b=\overline{v}_b) \\
=& \frac{P(b(v_b,t)=\overline{v}_b, v_b=\overline{v}_b, v_s=\underline{v}_s)}{P(v_b=\overline{v}_b, v_s=\underline{v}_s)} \overline{p} \\
=& \left[\frac{P(b(v_b,t)=\overline{v}_b, v_b=\overline{v}_b, t=\overline{t}, v_s=\underline{v}_s)}{P(v_b=\overline{v}_b, v_s=\underline{v}_s)} + \frac{b(v_b,t)=\overline{v}_b, v_b=\overline{v}_b, t=\underline{t}, v_s=\underline{v}_s}{P(v_b=\overline{v}_b, v_s=\underline{v}_s)}\right] \overline{p} \\
=& \left[\frac{P(b(v_b,t)=\overline{v}_b, v_b=\overline{v}_b, t=\overline{t}, v_s=\underline{v}_s)}{P(v_b=\overline{v}_b, t=\overline{t}, v_s=\underline{v}_s)} \times \frac{P(v_b=\overline{v}_b, t=\overline{t}, v_s=\underline{v}_s)}{P(v_b=\overline{v}_b, v_s=\underline{v}_s)}\right. \\
& \left. + \frac{P(b(v_b,t)=\overline{v}_b, v_b=\overline{v}_b, t=\underline{t}, v_s=\underline{v}_s)}{P(v_b=\overline{v}_b, t=\underline{t}, v_s=\underline{v}_s)} \times \frac{P(v_b=\overline{v}_b, t=\underline{t}, v_s=\underline{v}_s)}{P(v_b=\overline{v}_b, v_s=\underline{v}_s)}\right] \overline{p}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[P(b(v_b,t)=\overline{v}_b|v_b=\overline{v}_b, t=\overline{t}, v_s=\underline{v}_s) \times P(t=\overline{t}|v_b=\overline{v}_b, v_s=\underline{v}_s)\right. \\
&\left. + P(b(v_b,t)=\overline{v}_b|v_b=\overline{v}_b, t=\underline{t}, v_s=\underline{v}_s) \times P(t=\underline{t}|v_b=\overline{v}_b, v_s=\underline{v}_s)\right] \overline{p} \\
&= \left[P(b(\overline{v}_b,\overline{t})=\overline{v}_b) \times P(t=\overline{t}|v_s=\underline{v}_s) + P(b(\overline{v}_b,\underline{t})=\overline{v}_b) \times P(t=\underline{t}|v_s=\underline{v}_s)\right] \overline{p}.
\end{aligned}
$$

where the second equality follows from the fact that v_b and v_s are independent, the third equality follows from the fact that $P(b(v_b,t)=\overline{v}_b, v_b=\underline{v}_b, v_s=\underline{v}_s)=0$ (i.e., $\underline{v}_b$-type buyer will never report $\overline{v}_b$), and the last equality follows from the fact that v_b and t are independent.

Appendix 2

In this appendix, we will show that $P(\bar{t}|\overline{v}_s) > P(\bar{t}|\underline{v}_s)$. It is sufficient to show that $P(\underline{t}|\underline{v}_s) > P(\underline{t}|\overline{v}_s)$.

Notice that $P(\underline{v}_s) = P(\underline{v}_s|\underline{t})P(\underline{t}) + P(\underline{v}_s|\bar{t})P(\bar{t}) < P(\underline{v}_s|\underline{t})$. That is, we have $P(\underline{v}_s) < \frac{P(\underline{v}_s, \underline{t})}{P(\underline{t})}$. Thus, $P(\underline{t}) < \frac{P(\underline{v}_s, \underline{t})}{P(\underline{v}_s)}$, i.e., $P(\underline{t}) < P(\underline{t}|\underline{v}_s)$. Since $P(\underline{t}) = P(\underline{t}|\underline{v}_s)P(\underline{v}_s) + P(\underline{t}|\overline{v}_s)P(\overline{v}_s)$, we thus have $P(\underline{t}|\underline{v}_s) > P(\underline{t}|\overline{v}_s)$.

Appendix 3

In this appendix, we will show that a sufficient condition for $\overline{m}$ and $\underline{m}$ being sufficiently small is that $\overline{v}_b - \overline{v}_s$ and $\underline{v}_b - \underline{v}_s$ are sufficiently small.

For any given $\epsilon_1 \in (0, \frac{1}{2})$ and $\epsilon_2 \in (0, 2)$, define $\delta_1 = \epsilon_1 \frac{2 - \epsilon_2}{1 - \epsilon_1 \epsilon_2}$ and $\delta_2 = \epsilon_2 \frac{1 - 2\epsilon_1}{1 - \epsilon_1 \epsilon_2}$. We will show that if $\overline{v}_b - \overline{v}_s < \delta_1$ and $\underline{v}_b - \underline{v}_s < \delta_2$, then we must have $\overline{m} < \epsilon_1$ and $\underline{m} < \epsilon_2$. This follows from the following two facts. First, $\overline{m} = \frac{\overline{v}_b - \overline{v}_s}{2\overline{v}_b - \underline{v}_b - \underline{v}_s} = \frac{\overline{v}_b - \overline{v}_s}{2(\overline{v}_b - \underline{v}_s) - (\underline{v}_b - \underline{v}_s)} = \frac{\overline{v}_b - \overline{v}_s}{2 - (\underline{v}_b - \underline{v}_s)} < \frac{\delta_1}{2 - \delta_2} = \epsilon_1$. Second, $\underline{m} = \frac{\underline{v}_b - \underline{v}_s}{\underline{v}_b + \overline{v}_s - 2\underline{v}_s} = \frac{\underline{v}_b - \underline{v}_s}{\underline{v}_b - \underline{v}_s + \overline{v}_s - \underline{v}_s} < \frac{\underline{v}_b - \underline{v}_s}{\overline{v}_s - \underline{v}_s} = \frac{\underline{v}_b - \underline{v}_s}{\overline{v}_b - \underline{v}_s - (\overline{v}_b - \overline{v}_s)} < \frac{\delta_2}{1 - \delta_1} = \epsilon_2$.

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